ABOUT STEFAN’S DEFINITION OF A FOLIATION WITH SINGULARITIES: A REDUCTION OF THE AXIOMS

BY

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The following definitions of a foliation with singularities comes from the work by P. STEFAN [4].

Suppose $V$ is a connected Hausdorff $C^\infty$ and paracompact (equivalently and with a countable basis) manifold of dimension $n$. By a foliation of $V$ with singularities we mean a partition $\mathcal{F}$ of $V$ into sets such that:

1. For each element $L \in \mathcal{F}$, there exists a structure of differentiable manifold $\sigma$ on $L$ such that
   1. $(L, \sigma)$ is a connected immersed submanifold of $V$,
   2. $(L, \sigma)$ is a leaf of $V$ with respect to all locally connected topological spaces, i.e. if $X$ is an arbitrary locally connected topological space and $f: X \to V$ is a continuous function such that $f[X] \subset L$, then $f: X \to (L, \sigma)$ is continuous;

2. For each $x \in V$, there exists a local chart $\varphi$ on $V$ around $x$ with the following properties:
   1. $\varphi$ is a surjection $D_\varphi \to U_\varphi \times W_\varphi$ where $U_\varphi$, $W_\varphi$ are open neighbourhoods of 0 in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, respectively, and $k$ is the dimension of the leaf through $x$ (denoted by $L_x$);
   2. $\varphi(x) = (0, 0)$;

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(c) if \( L \in \mathcal{F} \), then \( \varphi[L \cap D_\varphi] = U_\varphi \times \ell_{\varphi,L} \) where

\[
\ell_{\varphi,L} = \{ w \in W_\varphi : \varphi^{-1}(0, w) \in L \}.
\]

A chart \( \varphi \) which fulfils the above condition is called distinguished around \( x \).

**Theorem.** — Let \( \mathcal{F} \) be a partition of \( V \) into connected immersed submanifolds of \( V \), fulfilling (2). Then \( \mathcal{F} \) is a foliation with singularities.

**Remark.** — This theorem is formulated in [3, p. 45] without an accurate proof. The author say that it easily follows in the same way as in the case without singularities, indicating [1]. It turns out that this theorem needs a subtler proof. The reasoning as in [1] gives the proof provided some added assumption

\[ (1) \quad \frac{\partial}{\partial \varphi^i} \bigg|_y \in T_y(L_x) \text{ for } i \leq k \text{ and all } y \in L_x \cap D_\varphi, \quad k = \dim L_x, \]

is satisfied, which is exactly the body of Stefan's lemma [4, Lemma 3.1]. That this added condition follows from the remaining ones is the aim of our paper.

**Proof of the Theorem:** according to Stefan ([4, Lemma 3.1]), it is sufficient to show that each distinguished chart \( \varphi = (\varphi^1, \ldots, \varphi^n) \) around \( x \) has the property (1).

Assume to the contrary that, for a distinguished chart \( \varphi \) around \( x \), this property does not hold at a point \( y_0 \in L_x \cap D_\varphi \). Then, of course, there exists a vector \( v \in T_{y_0}(L_x) \) such that

\[ (2) \quad \varphi_*v_0(v) \notin T_{(\tilde{y}_0, c_0)}(U_\varphi \times \{ c_0 \}) \]

where \((\tilde{y}_0, c_0) = \varphi(y_0), \tilde{y}_0 \in U_\varphi, c_0 \in W_\varphi\).

Take any smooth curve

\[ c : (-\varepsilon, \varepsilon) \rightarrow L_x, \quad \varepsilon > 0, \]

such that \( c(0) = y_0, \dot{c}(0) = v \) and \( \text{Im } c \subset D_\varphi \). Consider the curve \( \varphi \circ c : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n \). Let \( \text{pr}_2 : U_\varphi \times W_\varphi \rightarrow W_\varphi \) denotes the projection onto the second factor. By (2):

\[ (\text{pr}_2)_*((\varphi \circ c)(0)) \neq 0. \]
Diminishing $\varepsilon > 0$, if necessary, we may assume that
\[ pr_2 \circ \varphi \circ c : (-\varepsilon, \varepsilon) \to W_\varphi \]
is an embedding. Denote the set $\text{Im}(pr_2 \circ \varphi \circ c)$ by $I$. Of course,
\[ U_\varphi \times I \subset U_\varphi \times \ell_\varphi, L_x \]
(because $I \subset pr_2 \circ \varphi[D_\varphi \cap L_x] = \ell_\varphi, L_x$) and $U_\varphi \times I$ is a $k + 1$-dimensional hypersurface of $\mathbb{R}^n$, thus a locally compact space. Put — for brevity —
\[ M := L_x \cap D_x \]
understanding it as an open submanifold of $L_x$, and consider the injective immersion
\[ \tilde{\varphi} : M \to \mathbb{R}^n, \quad x \mapsto \varphi(x). \]
By the above $\tilde{\varphi}[M] \supset U_\varphi \times I$.

For each point $x \in M$, we choose a neighbourhood $U(x) \subset M$ of $x$ such that
\[ \tilde{\varphi}|_{U(x)} : U(x) \to \mathbb{R}^n \]
is an embedding. By the assumption of the second axiom of countability of $V$, each connected immersed submanifold of $V$ fulfils this axiom (see Appendix). Then $M$, as an open submanifold of the manifold $L_x$, has a countable basis. Choose a countable open covering $\{U_i ; i \in \mathbb{N}\}$ of $M$ such that each $U_i$ is compact and contained in some $U(x_i)$. We prove that
\[ \tilde{\varphi}[U_i] \cap (U_\varphi \times I) \]
— as a subset of the space $U_\varphi \times I$ — has no interior. We have a little more, namely that the set $\tilde{\varphi}[U(x_i)] \cap (U_\varphi \times I)$ has no interior. If it were not, then by taking an nonempty and open subset $X \subset U_\varphi \times I$ such that $X \subset \tilde{\varphi}[U(x_i)]$, we would obtain the mapping
\[ (\tilde{\varphi}|_{U(x_i)})^{-1}|_X : X \to U(x_i) \]
from a $(k + 1)$-dimensional manifold to a $k$-dimensional one, being an immersion, which is not possible. Thus $U_\varphi \times I$ is an union of a countable sequence of nowhere dense sets
\[ \{\tilde{\varphi}[U_i] \cap (U_\varphi \times I) ; i \in \mathbb{N}\}, \]
which leads to a contradiction with Baire's theorem for locally compact spaces. The theorem is proved.
Appendix: The following theorem is well known; here we give a simple proof of it.

**Theorem.** — Each connected immersed submanifold $L$ of a $C^\infty$ Hausdorff paracompact manifold $V$ has a countable basis.

Proof. — Let $f : L \to V$ be an immersion. The assumptions imply the existence of a Riemann tensor $G$ on $V$. Its pullback $f^*G$ is a Riemann tensor on $L$. A connected manifold which possesses a Riemann tensor is separable [2], therefore it has a countable basis. 

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**BIBLIOGRAPHIE**