Linear direct connections in Lie groupoids, curvature and characteristic classes

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The plan of the talk

1. Linear direct connections $\tau$ (called also linear quasi-connections) in tangent bundles and in vector bundles. The Teleman’s theorem
2. Underlying a usual linear connection $\nabla^\tau$ and a direct proof of this theorem, the curvature of $\tau$ versus connection of $\nabla^\tau$.
3. Groupoids point of view and groupoids generalizations.
1 Linear direct connections in vector bundles and Teleman’s theorem

Nicola Teleman in the papers

N.Teleman, Distance Function, Linear quasi-Connections and Chern Character, June 2004, IHES/M/04/27

N.Teleman, Direct Connections and Chern Character, Proceedings of the International Conference in Honor of Jean-Paul Brasselet, Luminy, May 2005,

shows how the Chern character of the tangent bundle of a smooth manifold may be extracted from the geodesic distance function by means of cyclic homology.
The processing has the following steps:

1. Let $M$ be a smooth Riemannian manifold and let

   \[ r : M \times M \to [0, \infty) \]

be the induced geodesic distance function.

   The function $r^2$ is smooth on a neighbourhood of the diagonal.

2. Let $\chi$ be a cut-off smooth monotone decreasing real valued function, identically 1 on a neighbourhood of 0, having support on a sufficiently small interval, so that $\chi \circ r^2$ be well defined and smooth. For $x, y \in M$ a linear mapping

   \[ A(y, x) : T_xM \to T_yM \]

is given by the formula

   \[ A(y, x) \left( \sum_i \xi^i \frac{\partial}{\partial x^i} \right) = \sum_{i,j,k} \xi^i \frac{\partial^2 (\chi \circ r^2)(x, y)}{\partial x^i \partial y^j} g^{jk}(y) \frac{\partial}{\partial y^k} \]

$(A(y, x)$ is independent of the local coordinates).

For sufficiently close points $x, y$,

- $A(y, x)$ is an isomorphism and
- $A(x, x)$ is the identity.

Therefore $A$ is a linear direct connection (=linear quasi-connection), with respect to the definition below.
3. With the object $A$ there is associated the function $\Phi_k : U_{k+1} \to \mathbb{R}$, where $U_{k+1}$ is a neighbourhood of the diagonal in $M^{k+1}$

$$\Phi_k (x_0, x_1, \ldots, x_k) := \text{Trace} \ A(x_0, x_1) \circ A(x_1, x_2) \circ \cdots \circ A(x_{k-1}, x_k) \circ A(x_k, x_0).$$

4. Next, N. Teleman studies the function $\Phi_k$ in the context of cyclic homology:

— firstly, he notices that $\Phi_k$, $k$ = even, is a cyclic cycle over the algebra $\mathcal{A} = C^\infty (M)$,

— secondly, he uses the Connes’ isomorphism which associates with $\Phi_k$ a closed differential form

$$\Omega (\Phi_k) (x) = \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k} \frac{\partial}{\partial x_1^{i_1}} \frac{\partial}{\partial x_2^{i_2}} \cdots \frac{\partial}{\partial x_k^{i_k}} \Phi_k (x_0, x_1, \ldots, x_k)_{x_0=x_1=\cdots=x_k=x} \ dx_1^{i_1} \wedge \cdots \wedge dx_k^{i_k},$$

we use the same local coordinate system on each factor).

— thirdly, he proves

**Theorem 1** The top degree component of the cyclic homology class of $\Phi_k$ is equal to

$$[\Omega (\Phi_k)] = c \cdot Ch_k (M)$$

where $c$ is a constant and $Ch_k (M)$ is the $k$-component of the Chern character of the tangent bundle of $M$. 

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The object $A$ is a particular case of the linear direct connection introduced by N. Teleman.

**Definition 2** Let $E$ be a real or complex smooth vector bundle over the manifold $M$. A **linear direct connection** $\tau$ in $E$ consists of assigning to any two points $x, y \in M$, sufficiently close one to each other, an isomorphism

$$\tau(y, x) : E|_x \to E|_y,$$

such that

$$\tau(x, x) = id,$$

and $\tau(y, x)$ depends smoothly on the pair $x, y$. 

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The parallel transport defined by a usual linear connection in $E$ along the small geodesics of an affine connection in $M$ induces a linear direct connection in $E$ (see for example A.Connes and H.Moscovici, "Cyclic cohomology, the Novikov conjecture and hyperbolic groups", Topology 29, n 3 345-388, 1990).

- i) As for $A$ with $\tau$ there is associated the function $\Phi_k$ by the formula $$\Phi_k (x_0, x_1, ..., x_k) := \text{Trace} \, \tau (x_0, x_1) \circ \tau (x_1, x_2) \circ ... \circ \tau (x_{k-1}, x_k) \circ \tau (x_k, x_0).$$

The function $$\Phi_2 (x_0, x_1, x_2) = \text{Trace} \, \tau (x_0, x_1) \circ \tau (x_1, x_2) \circ \tau (x_2, x_0)$$ plays a role of the curvature of $\tau$ and the differential form $\Omega (\Phi_2)$ - the curvature form of $\tau$.

- ii) Any two smooth linear direct connections in a smooth vector bundle are smoothly homotopic. The results above imply

**Theorem 3 (N.Telean)** For any smooth linear direct connection $\tau$ in the smooth vector bundle $E$ over the manifold $M$,

- i) $\Phi_k$, $k = \text{even}$, is a cyclic cycle over the algebra $C^\infty (M)$,

- ii) the cohomology class of $\Omega (\Phi_{2k})$ is (up to a multiplicative constant) is the $k$-component of the Chern character of $E$.  

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2 Underlying linear connection $\nabla^\tau$ and a direct proof of this theorem

In the paper J.Kubarski, N.Teleman, *Linear direct connections*, Banach Center Publications, 2007, in print, we study the geometry of direct connections $\tau$:

- we construct the "infinitesimal part" $\nabla^\tau$ and show that $\nabla^\tau$ is a usual linear connection. We next determine the curvature tensor $R$ of $\nabla^\tau$ and show that the equality of differential forms holds

\[ \Omega(\Phi_k) = c \cdot Tr R^k. \]
We intend to extract from a direct connection its infinitesimal part along the diagonal.

**Definition 4** Let $X$ be a smooth tangent field over $M$ and $\phi$ a smooth section in $E$. Let $x_0$ be an arbitrary point in $M$ and let $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ be an integral path of the field $X$ with the initial condition $\gamma(0) = x_0$.

We define

$$\nabla^r_{X(x_0)}(\phi) = \frac{d}{dt} \{\tau(\gamma(0), \gamma(t)) (\phi(\gamma(t)))\}_{t=0} \in E|_{x_0}.$$ 

**Theorem 5** The right hand side of the above formula depends only on the value of $X$ at $x_0$. The operator $\nabla^r_{X(x_0)}(\phi)$ is a usual linear connection in $E$.

We intend to describe $\nabla^r_{X(x_0)}(\phi)$ locally.
Let \((x^1, x^2, ..., x^m) (\dim M = m)\) be a local coordinate system on an open neighborhood \(V\) of a point \(x_0\). Using the same local coordinate system on both factors of the direct product \(M \times M\), any point \((x, y) \in V \times V\) will be given by local coordinates \((x^1, x^2, ..., x^m|y^1, y^2, ..., y^m)\).

**Theorem 6** Let \(\{e_1, e_2, ..., e_n\}\) be a local frame in \(E\) over \(V\). Let \(\tau(x|y)\) be the matrix describing locally the direct connection \(\tau\):

\[
\tau(x|y) = ||\tau^j_i(x|y)|| \in M_{n,n}(\mathbb{K}),
\]

\[
\tau(x, y)(e_i(y)) = \sum_j \tau^j_i(x|y) \cdot e_j(x), \quad \tau^j_i(x|x) = \delta^j_i.
\]

Then the coefficients \(\Gamma^j_{i,\alpha}\) of the connection \(\nabla^\tau\) are given locally by

\[
\nabla^\tau_{\partial x^\alpha} e_i = \sum_j \Gamma^j_{i,\alpha} e_j,
\]

where

\[
\Gamma^j_{i,\alpha}(x) = \frac{\partial}{\partial y^\alpha} \tau^j_i(x^1, x^2, ..., x^m|y^1, y^2, ..., y^m)_{y=x}.
\]

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In conclusion, representing the tangent field \( X \) locally

\[
X(x) = \sum_{\alpha} X^\alpha(x) \cdot \frac{\partial}{\partial x^\alpha},
\]

one has the formula

\[
\nabla^r_{X(x_0)}(\sum_i \phi^i e_i) = \sum_{\alpha=1}^m \left\{ \sum_{i,j} \Gamma^j_{i,\alpha}(x_0) \cdot X^\alpha(x_0) \cdot \phi^i(x_0) e_j(x_0) \right\} + \sum_i (d\phi^i)(X(x_0)) e_i(x_0).
\]

**Remark 7** The above formula also show that \( \nabla^r \) is a linear connection in the vector bundle \( E \). The linear connection \( \nabla^r \) will be called associated, or underlying, linear connection to the direct connection \( \tau \).
Proposition 8 Let \( R = (\nabla^r)^2 \) be the curvature tensor of the connection \( \nabla^r \).
The components of the curvature \( R \) are

\[
R^i_{\alpha\beta}(x) = \frac{\partial}{\partial x^\alpha} \Gamma^i_{\beta\gamma}(x) - \frac{\partial}{\partial x^\beta} \Gamma^i_{\alpha\gamma}(x) + \Gamma^i_{\alpha\lambda}(x) \cdot \Gamma^\lambda_{\beta\gamma}(x) - \Gamma^i_{\beta\lambda}(x) \cdot \Gamma^\lambda_{\alpha\gamma}(x)
\]

\[
= \frac{\partial^2}{\partial x^\alpha \partial y^\beta} \tau^i_j(x|y)_{y=x} + \frac{\partial^2}{\partial x^\beta \partial y^\alpha} \tau^i_j(x|y)_{y=x} + \frac{\partial}{\partial y^\alpha} \tau^i_k(x|y)_{y=x} \cdot \frac{\partial}{\partial y^\beta} \tau^k_i(x|y)_{y=x} - \frac{\partial}{\partial y^\beta} \tau^i_k(x|y)_{y=x} \cdot \frac{\partial}{\partial y^\alpha} \tau^k_i(x|y)_{y=x}.
\]

Corollary 9 The curvature form \( R \) of the underlying linear connection \( \nabla^r \), associated to the direct connection \( \tau \), is given by

\[
R = \left( \frac{\partial^2}{\partial x^\alpha \partial y^\beta} \tau^i_j(x|y)_{y=x} - \frac{\partial^2}{\partial x^\beta \partial y^\alpha} \tau^i_j(x|y)_{y=x} + \frac{\partial}{\partial y^\alpha} \tau^i_k(x|y)_{y=x} \cdot \frac{\partial}{\partial y^\beta} \tau^k_i(x|y)_{y=x} - \frac{\partial}{\partial y^\beta} \tau^i_k(x|y)_{y=x} \cdot \frac{\partial}{\partial y^\alpha} \tau^k_i(x|y)_{y=x} \right) dx^\alpha \wedge dx^\beta.
\]
Although, \( \tau(x, y) = (\tau(y, x))^{-1} \) is not true in general, it is true, however, that it holds infinitesimally. In fact, we have the

**Proposition 10** For any direct connection \( \tau \), its matrix components satisfy the identities

-\( i) \) \[ \frac{\partial}{\partial x^\alpha} \tau^j_i(x|y)_{y=x} + \frac{\partial}{\partial y^\alpha} \tau^j_i(x|y)_{y=x} = 0. \]

-\( ii) \) \[ \frac{\partial}{\partial x^\alpha} \{ \tau(x|y) \circ \tau(y|x) \}_{y=x} = 0 = \frac{\partial}{\partial y^\alpha} \{ \tau(x|y) \circ \tau(y|x) \}_{y=x}. \]

As \( \tau(x|x) = Id. \), we get that the directional derivative \( (\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) \) of \( \tau \) along the diagonal vanishes. This proves -i). The second identity is a consequence of the first.

The above properties of any direct connection are fundamental for comparing the curvature tensor \( R \) to the differential form \( \Omega (\Phi^r_{2k}) \). 

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We obtain an important explicit link between $\Omega(\Phi_{2k})$ and the classical Chern-Weil forms, at the level of differential forms rather than cohomology classes.

**Theorem 11** Let $\tau$ be a direct connection and let $\nabla^\tau$ be its underlying linear connection. Then

$$\Omega(\Phi^\tau_2) = \frac{1}{4} \cdot Tr \, R,$$

and more generally,

$$\Omega(\Phi^\tau_{2k}) = \frac{1}{(2k)!} \cdot \frac{1}{2k} \cdot Tr \, R^k,$$

where $R = (\nabla^\tau)^2$ is the curvature of the underlying linear connection $\nabla^\tau$.

In consequence, the mentioned above Teleman’s theorem follows from this directly.
3  Groupoids point of view and groupoids generalizations

N. Teleman in your papers said:
"The arguments discussed here may be extended to the language of groupoids".

My further talk is the first step in this direction.

3.1 Direct connections and the Lie groupoid $GL(E)$

Let $E$ be a real or complex smooth vector bundle over the manifold $M$. Consider the transitive Lie groupoid

$$\Phi = GL(E)$$

of all linear fibre isomorphisms $h : E_x \to E_y$ of the vector bundle $E$, with the source $\alpha$, $\alpha(h) = x$, and the target $\beta$, $\beta(h) = y$, and the unit $u_y = \text{id}_{E_y}$.

The mappings

$$\alpha, \beta : \Phi \to M, \quad (\alpha, \beta) : \Phi \to M \times M$$

are submersions, the injection

$$u : M \to \Phi, \quad y \to u_y,$$

is smooth, and the partial multiplication

$$\cdot : \Phi \times_{(\alpha, \beta)} \Phi \to \Phi, \quad (g, h) \mapsto gh,$$

is also smooth. and $GL(E)$ be a Lie groupoid of linear fibre isomorphisms.
Remark 12. A linear direct connection in a vector bundle $E$ is equivalently a smooth mapping

$$\tau : U \to GL(E)$$

where $U \subset M \times M$ is an open neighborhood of the diagonal $\Delta = \{(x, x) ; x \in M\}$, such that

$$\tau(x, y) : E|_y \to E|x$$

i.e.

$$\alpha \circ \tau(x, y) = y, \quad \beta \circ \tau(y, x) = x,$$

and

$$\tau(x, x) = id : E|x \to E|x.$$
3.2 Lie Groupoids and point of view of linear direct connections and the using of the Lie algebroids

According to the Pradines definition, the Lie algebroid of an arbitrary transitive Lie groupoid $\Phi$ is equal to the vector bundle

$$A(\Phi) = u^* (T^\alpha \Phi)$$

where $u : M \to \Phi$, $y \to u_y$, and $T^\alpha \Phi = \ker \alpha_*$, equipped with the suitable structures: the bracket of cross-sections $[[\xi, \eta]]$, $\xi, \eta \in \text{Sec} A(\Phi)$ is defined in the following way. The cross-sections $\xi, \eta$ can be extended to right invariant vector fields $\xi', \eta'$ on $\Phi$, their usual bracket $[[\xi', \eta']]$ is invariant too, so it determines a cross-section of $u^* (T^\alpha \Phi)$ denoting by $[[\xi, \eta]]$. The anchor is defined as the restriction of $\beta_*$. 

We recall the definition of a Lie algebroid.
Definition 13  By a Lie algebroid on a manifold $M$ we mean a system

$$A = (A, [\cdot, \cdot], \gamma)$$

consisting of a vector bundle $A$ (over $M$) and mappings

$$[\cdot, \cdot] : \text{Sec}
A \times \text{Sec}
A \rightarrow \text{Sec}
A,
\gamma : A \rightarrow TM,$$

such that

(i) $(\text{Sec}
A, [\cdot, \cdot])$ is an $\mathbb{R}$-Lie algebra,

(ii) $\gamma$, called an anchor, is a homomorphism of vector bundles,

(iii) $\text{Sec}
\gamma : \text{Sec}
\gamma \rightarrow \mathfrak{X}
(M), \xi \mapsto \gamma \circ \xi$, is a homomorphism of Lie algebras,

(iv) $[\xi, f \cdot \eta] = f \cdot [\xi, \eta] + (\gamma \circ \xi)(f) \cdot \eta$ for $f \in C^\infty(M), \xi, \eta \in \text{Sec}
A$.

Lie algebroid (1) is called transitive if $\gamma$ is an epimorphism. $g = \ker \gamma$ is a vector bundle, called the adjoint of (1), and the short exact sequence

$$0 \rightarrow g \xleftarrow{\gamma} A \xrightarrow{\gamma} E \rightarrow 0$$

is called the Atiyah sequence of (1).
Example 14 The following are simple fundamental examples of transitive Lie algebroids:

(1°) Finitely dimensional Lie algebra.

(2°) Tangent bundle $TM$ to a manifold $M$ with the bracket $[,]$ of vector fields and $id_{TM}$ as an anchor.

(3°) Trivial Lie algebroid $TM \times \mathfrak{g}$ (Ngo-Van-Que) where $\mathfrak{g}$ is as in (1°). The bracket is defined by the formula,

$$\llbracket (X, \sigma), (Y, \eta) \rrbracket = ([X, Y], \mathcal{L}_X \eta - \mathcal{L}_Y \sigma + [\sigma, \eta]),$$

$X, Y \in \mathfrak{X}(M)$, $\sigma, \eta : M \to \mathfrak{g}$, and the anchor is the projection $TM \times \mathfrak{g} \to TM$.

(4°) Bundle of jets $J^kTM$ (P.Libermann).
General form (K.Mackenzie, J.Kubarski). Let a system \((g, \nabla, \Omega_b)\) be given, consisting of a Lie algebra bundle \(g\) on a manifold \(M\), a covariant derivative \(\nabla\) in \(g\) and a 2-form \(\Omega_b \in \Omega^2(M, g)\) on \(M\) with values in \(g\), fulfilling the conditions:

\[(i)\] \[\nabla^2 \sigma = -[\Omega_b, \sigma], \quad \sigma \in \text{Sec} \ g,\]

\[(ii)\] \[\nabla_X [\sigma, \eta] = [\nabla_X \sigma, \eta] + [\sigma, \nabla_X \eta], \quad X \in X(M), \quad \sigma, \eta \in \text{Sec} \ g,\]

\[(iii)\] \[\nabla \Omega_b = 0.\]

Then \(TM \oplus g\) forms a transitive Lie algebroid with the bracket defined by

\[\langle (X, \sigma), (Y, \eta) \rangle = \langle [X, Y], -\Omega_b(X, Y) + \nabla_X \eta - \nabla_Y \sigma + [\sigma, \eta]\rangle,
\]
the anchor being the projection onto the first component.

Every transitive Lie algebroid is — up to an isomorphism — of this form.
Example 15 The following are important examples of transitive Lie algebroids:


(7) The Lie algebroid $CDO(E)$ of covariant differential operators on a vector bundle $E$ (K.Mackenzie). Another isomorphic construction of this object is the Lie algebroid $A(E)$ of a vector bundle $E$ (J.Kubarski), here the fibre $A(E)|_x$ is the space of linear homomorphisms $l : \text{Sec } E \to E|_x$ such that there exists a vector $u \in T_xM$ for which $l(f \cdot \nu) = f(x) \cdot l(\nu) + u(f) \cdot \nu(x)$, $f \in C^\infty(M)$, $\nu \in \text{Sec}(E)$.

(8) The Lie algebroid $A(\Phi) := i^*T^\alpha \Phi$ of a Lie groupoid $\Phi$ (J.Pradines). (Remark: if $\Phi = GL(E)$ is the Lie algebroid of all linear fibre isomorphisms of fibres of $E$ then $A(E) = A(\Phi)$).

(9) The Lie algebroid $A(M, F)$ of a transversally complete foliation $(M, F)$ (P.Molino); in particular,

(10) the Lie algebroid $A(G; H)$ of the foliation of left cosets of a Lie group $G$ by a nonclosed connected Lie subgroup $H \subset G$ (for the construction independent of the theory of transversally complete foliations, see J.Kubarski).
There are many sources of nontransitive Lie algebroids: Lie equations, Differential groupoids, Poisson manifolds, etc.
Let $\Phi = GL(E)$ be the Lie groupoid of all linear fibre isomorphisms of fibres of $E$.

For $y \in M$ the submanifold $\Phi_y = GL(E)_y \subset GL(E)$ of all elements $u \in GL(E)$ for which $\alpha(u) = y$,

$$GL(E)_y = \alpha^{-1}(y),$$

is a $GL(E_y)$-principal fibre bundle.

- Lie algebroid of the Lie groupoid is the infinitesimal object and play analogous role to that of Lie algebras for Lie groups.

- The space [Lie algebra] of global cross-sections $Sec(A(\Phi)), \Phi = GL(E)$ where $E$ is a vector bundle, is naturally isomorphic to the Lie algebra of all Covariant Derivative Operators, i.e. to the space of differential operators of the rank $\leq 1$

$$\mathfrak{L} : SecE \to SecE$$

such that $\mathfrak{L}(f \cdot \xi) = f \cdot \mathfrak{L}(\xi) + X(f) \cdot \xi$, for a vector field $X$ called the anchor of $\mathfrak{L}$, $f \in C^\infty(M), \xi \in SecE$. 

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Let \( \tau : (M \times M | U \rightarrow GL(E) \) (where \( U \subset M \times M \) is an open neighbourhood of the diagonal \( \Delta = \{(x, x) : x \in M\} \)) be a linear quasi-connection,

\[
\tau_{(x, y)} : E|_y \rightarrow E|x,
\]

so \( \alpha(\tau_{(x, y)}) = y \) and \( \beta(\tau_{(x, y)}) = x \) and let \( \nabla^\tau \) be the underlying linear connection of \( \tau \) in \( E \).

Now, we fix \( y \) and take

\[
\tau(\cdot, y) : M \rightarrow GL(E)_y, \quad x \mapsto \tau(x, y).
\]

It is a smooth mapping such that \( \beta \circ \tau(\cdot, y) = \text{id} \). Therefore the composition of the differential

\[
\tau(\cdot, y)_x : T_x M \rightarrow T_{\tau(x, y)} \left(GL(E)_y\right)
\]

with the differential of \( \beta|GL(E)_y \rightarrow M \) is identity

\[
\text{id} : T_x M \xrightarrow{\tau(\cdot, y)_x} T_{\tau(x, y)} \left(GL(E)_y\right) \xrightarrow{\beta_*} T_x M.
\]

Taking \( x = y \) and using the fact \( \tau(y, y) = u_y = \text{id}_{E_y} \) we see that

\[
\tau(\cdot, y)_y : T_y M \rightarrow T_{u_y} \left(GL(E)_y\right).
\]
Therefore $\tau$ determines a usual connection

$$\nabla^\tau : TM \to u^*(T^\alpha \Phi)$$

$$\nabla^\tau (v_y) = \tau(y, y)_y (v_y).$$

in the Lie algebroid $u^*(T^\alpha \Phi) (\Phi = GL(E))$, i.e. a splitting of the Atiyah sequence

$$0 \to g \to A(\Phi) \xrightarrow{\nabla^\tau} TM \to 0.$$ 

$\nabla^\tau$ is the "usual covariant derivative" since the anchor of the Covariant Derivative Operator $\nabla^\tau (X) : \text{Sec} E \to \text{Sec} E$ is just equal to $X$, therefore noticing $\nabla^\tau (X) (\xi)$ in the form

$$\nabla^\tau_X (\xi)$$

the usual axioms for covariant derivative are fulfilled.

**Theorem 16** $\nabla^\tau = \nabla^\tau$, i.e. the connection $\nabla^\tau$ is equal to the underlying linear connection of $\tau$ in $E$. 

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Proof. (a sketch) Since we need to prove it at any point \( y \in M \) so we can prove it locally for \( E = M \times \mathbb{R}^n \) and \( M = \mathbb{R}^m \). Then \( GL(E) = M \times GL(\mathbb{R}^n) \times M \), 
\[ \alpha^{-1}(y) = GL(E)_y = M \times GL(\mathbb{R}^n) \times \{y\} \]. Let \( \{e_i\}_{i=1}^n \) be a trivial local basis of \( E \), then the induced linear connection \( \nabla^\tau \) is determined by

\[
\nabla^\tau_{\frac{\partial}{\partial x^k}} e_i = \frac{\partial \tau_{ij}^k}{\partial x^{m+k}} (y, y) \cdot e_j.
\]

We can obtain the same results for \( \nabla^\tau \). \( \blacksquare \)
3.3 Groupoids generalization

The above consideration has "groupoids sense" so we can it generalize to any transitive Lie groupoids.

Let $\Phi$ be an arbitrary transitive Lie groupoid with the anchor $\alpha$ and the target $\beta$. We denote by $u_y$ the unit of $\Phi$ at $y$.

**Definition 17** By a linear direct connection in $\Phi$ we mean a mapping

$$\tau : (M \times M)|_U \to \Phi,$$

such that

$$\alpha \circ \tau (x, y) = y, \quad \beta \circ \tau (x, y) = x,$$

and

$$\tau (x, x) = u_x.$$

For $y$ the submanifold $\Phi_y \subset \Phi$ of all elements $h \in \Phi$ for which $\alpha (h) = y$ ($\Phi_y = \alpha^{-1} (y)$ ) is a $\Phi_y^o$-principal fibre bundle where

$$\Phi_y^o = \{ h \in \Phi; \alpha (h) = \beta (h) = y \}$$

is the isotropy Lie algebra of $\Phi$ at $y$. Now, we fix $y$ and take

$$\tau (\cdot, y) : M \to \Phi_y, \quad x \mapsto \tau (x, y).$$
It is a smooth mapping such that

\[ \beta \circ \tau (\cdot, y) = \text{id}. \]

Taking the differential

\[ \tau (\cdot, y)_{sx} : T_x M \to T_{\tau(x,y)} (\Phi_y) \]

such that the composition with the differential of \( \beta |_{\Phi_y} \to M \) is identity

\[ \text{id} : T_x M \xrightarrow{\tau (\cdot, y)_{sx}} T_{\tau(x,y)} (\Phi_y) \xrightarrow{\beta} T_x M \]

and taking \( x = y \) and using the fact \( \tau (y, y) = u_y \) we see that

\[ \tau (\cdot, y)_{sy} : T_y M \to T_{u_y} (\Phi_y) = A(\Phi)|_y, \]

where \( A(\Phi) \) is the Lie algebroid of the Lie groupoid \( \Phi \).
Therefore $\tau$ determines a splitting of the Atiyah sequence of $\Phi$

$$0 \to g \to A(\Phi) \xrightarrow{\beta^*} TM \to 0,$$

i.e. a usual connection in the Lie algebroid $A(\Phi) = u^* (T^\alpha \Phi)$,

$$\nabla^\tau : TM \to u^* (T^\alpha \Phi) = A(\Phi)$$

$$\nabla^\tau (v_y) = \tau (\cdot, y)_y (v_y)$$

The connection $\nabla^\tau$ will be called the **underlying linear connection of the linear direct connection** $\tau$. .
Now we can ask on a very important question:

- How can we reconstruct the curvature tensor of $\nabla$ from the linear direct connection in the Lie groupoid $\Phi$? And next how can we reconstruct the Chern-Weil homomorphism of Lie groupoids $\Phi$ (i.e. equivalently of the principal bundle $\Phi_y$) from arbitrary taken linear direct connection $\tau$?

### 3.4 Curvature tensor of the linear direct connection in transitive Lie groupoids

Take any transitive Lie groupoid $\Phi$ and its Lie algebroid $A(\Phi)$ with the Atiyah sequence

$$0 \to g \to A(\Phi) \to TM \to 0.$$ 

The fibre of $g$ at $x$

$$g_x = T_{u_x} \Phi^x_x$$

is the right Lie algebra of the structural Lie group $\Phi^x_x$. For a linear direct connection $\tau$ in $\Phi$ denote by

$$\nabla^\tau : TM \to A(\Phi)$$

the underlying linear connection in the Lie algebroid $A(\Phi)$ induced by $\tau$. Consider the curvature tensor $\Omega^\tau \in \Omega^2 (M; g)$ of $\nabla^\tau$

$$\Omega^\tau (X, Y) = [[\nabla^\tau X, \nabla^\tau Y]] - \nabla^\tau_{[X, Y]},$$
The linear direct connection $\tau$ determines the mapping

$$\Psi_k : \left( \frac{M \times \ldots \times M}{\underbrace{\ldots}} \right)_{k+1} |_{U} \rightarrow \Phi,$$

$$\Psi_k^r (x_0, x_1, \ldots, x_k) = \tau (x_0, x_1) \cdot \tau (x_1, x_2) \cdot \ldots \cdot \tau (x_{k-1}, x_k) \cdot \tau (x_k, x_0)$$

having the values in the associated **Lie group bundle**,

$$\Psi_k^r (x_0, x_1, \ldots, x_k) \in \Phi^x_{x_0}.$$ 

For example, for $k = 2$, the function

$$\Psi_2^r : (M \times M \times M)_{|U} \rightarrow \Phi,$$

$$\Psi_2^r (x_0, x_1, x_2) = \tau (x_0, x_1) \cdot \tau (x_1, x_2) \cdot \tau (x_2, x_0)$$

is called the *curvature* of $\tau$.

Analogously to the previous cases we can associate some **differential form** to the function $\Psi_k$. Namely, fixing a point $x_0$ we define

$$\Psi_k^r (x_0) : \left( \frac{M \times \ldots \times M}{\underbrace{\ldots}} \right)_{k} |_{U} \rightarrow \Phi^x_{x_0},$$

$$(x_1, \ldots, x_k) \mapsto \Psi_k^r (x_0, x_1, \ldots, x_k) \in \Phi^x_{x_0}.$$
Next, we take a coordinate system \((x^1, \ldots, x^m)\) \((\text{dim} \, M = m)\) on an open neighborhood \(V\) of the point \(x_0\). Using the same local coordinate system on each factors of the direct product \(M \times \cdots \times M\) we take for \((x_1, \ldots, x_k) \in V \times \cdots \times V\)

\[
\frac{\partial}{\partial x^j_k} \Psi^T_k (x_0) \in T_{\Psi_k(x_0,x_1,\ldots,x_k)} \Phi^x_{x_0}.
\]

This vector we can translate via \textbf{right} translation to the unit. Let \(r_h: \Phi^x_{x_0} \to \Phi^x_{x_0}\) denote the right translation on the element \(h\), \(r_h(z) = z \cdot h\).

\[
\frac{\partial}{\partial x^j_k} \Psi^T_k (x_0) := \left(\tau_{(\Psi_k(x_0,x_1,\ldots,x_k))}^{-1}\right)_{*\Psi_k(x_0,x_1,\ldots,x_k)} \left(\frac{\partial}{\partial x^j_k} \Psi^T_k (x_0)\right) \in T_u_{x_0} (\Phi^x_{x_0}) = g_{x_0}.
\]

The function obtained

\[
(x_1, \ldots, x_{k-1}, x_k) \mapsto \frac{\partial}{\partial x^j_k} \Psi^T_k (x_0) \in g_{x_0}
\]

can be differentiated usually as a vector valued function.

\[
(x_1, \ldots, x_{k-1}, x_k) \mapsto \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \cdots \frac{\partial}{\partial x^{i_{k-1}}} \frac{\partial}{\partial x^k} \Psi^T_k (x_0, x_1, \ldots, x_k) \in g_{x_0}.
\]

We put

\[
\Omega \left( \Psi^T_k \right) (x) = \frac{1}{k!} \sum_{i_1,i_2,\ldots,i_k} \frac{\partial}{\partial x^{i_1}} \frac{\partial}{\partial x^{i_2}} \frac{\partial}{\partial x^k} \Psi^T_k (x_0, x_1, \ldots, x_k) \mid_{x_0=x_1=\ldots=x_k=x} \, dx^{i_1} \wedge \ldots \wedge dx^{i_k}.
\]
It is a $k$-form on $M$ with values in the vector bundle $g$

$$\Omega (\Psi_k^\tau) \in \Omega^k (M; g)$$

Considering $k = 2$ we obtain a 2-form with values in $g$, 

$$\Omega (\Psi_2^\tau) \in \Omega^2 (M; g),$$
called the **curvature form** of $\tau$.

The fundamental role is playing by the following

**Theorem 18** For an arbitrary linear direct connection $\tau : (M \times M)_U \to \Phi$ in the Lie groupoid $\Phi$ the curvature form of $\tau$ and the curvature form of the underlying connection in $A(\Phi)$ are differs on a constant

$$\Omega (\Psi_2^\tau) = \frac{1}{4} \cdot \Omega^\tau.$$

**Proof.** (a sketch) three steps of the proof:

- 1) Of course, we need to prove the equality point by point, so we can look at this locally. Assume that $\Phi$ is a trivial Lie groupoid $\Phi = M \times G \times M$ with the source $pr_3$ and the target $pr_1$ and the partial multiplication

$$(z, a, y) \cdot (y, b, x) = (z, ab, x).$$

Then $A(\Phi)_{|x} = T_x M \times g$ where $g$ is the right! Lie algebra of $G$,

$$A(\Phi) = TM \times g.$$
The bracket in $g$ will be denoted by $[v, w]^R$. The linear direct connection $\tau$ is given by

$$\tau : (M \times M)_{|U} \to M \times G \times M$$
$$\tau(x, y) = (x, \hat{\tau}(x, y), y),$$
$$\hat{\tau} : (M \times M)_{|U} \to G, \quad \hat{\tau}(x, x) = e.$$

Therefore

$$\frac{\partial \hat{\tau}}{\partial x^i}_{|\tau(x, x)} + \frac{\partial \hat{\tau}}{\partial x^{m+i}}_{|\tau(x, x)} = 0$$

and

$$\Psi^\tau_2(x_0, x_1, x_2) = (x_0, \hat{\tau}(x_0, x_1) \cdot \hat{\tau}(x_1, x_2) \cdot \hat{\tau}(x_2, x_0), x_0).$$

The induced linear connection $\nabla^\tau : TM \to TM \times g$ is equal to

$$\nabla^\tau_x(v) = \nabla^r_y(v) = \tau(\cdot, y)_{*y}(v) = \left(v, \hat{\tau}(\cdot, y)_{*y}(v)\right).$$

We calculate the curvature tensor $\Omega^\tau$ of $\nabla^\tau$.

$$\Omega^\tau_{|x_0} = 2 \sum_{i<j} \left( \frac{\partial^2 \hat{\tau}}{\partial x^i \partial x^{m+i}} - \frac{\partial^2 \hat{\tau}}{\partial x^i \partial x^{m+j}} + \left[ \frac{\partial \hat{\tau}}{\partial x^i}, \frac{\partial \hat{\tau}}{\partial x^j} \right]^R \right)_{|(x_0, x_0)} dx^i \wedge dx^j.$$

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-ii) Now we calculate the form $\Omega (\Psi_2)$:

$$
\Omega (\Psi_2) (x_0) = \frac{1}{2!} \sum_{i,j} \left. \frac{\partial^2 \Psi_2 (x_0, x_1, x_2)}{\partial x_i \partial x_j} \right|_{x_0=x_1=x_2} \, dx^i \wedge dx^j
$$

for $\Psi_2 (x_0, x_1, x_2) = \hat{t} (x_0, x_1) \cdot \hat{t} (x_1, x_2) \cdot \hat{t} (x_2, x_0)$.

-a) Firstly we calculate it for the Lie group $G$ of matrices,

$$
G \subset GL (V) \subset \mathbb{R}^{(\text{dim} \, V)^2}
$$

for some finitely dimensional vector space $V$, using the fact that the differential of the left and the right translations, $l_g$ and $r_g$, are exactly respectively the left and the right multiplication by matrices,

$$(l_g)_{sh} = gh, \quad (r_g)_{sh} = hg.$$

After calculations we obtain the equality

$$
\Omega (\Psi_2) = \frac{1}{4} \cdot \Omega^r.
$$
-b) For an arbitrary Lie group $G$ we use the following "IMPORTANT LOCAL TRICK":

We need to consider only local Lie group structure near the unit. But every Lie algebra is isomorphic to a Lie algebra of matrices (because there exists a faithful representation in some finitely dimensional vector space $V$) and a Lie algebra of matrices is a Lie algebra of a Lie subgroup of the Lie group $GL(V)$. Therefore the above result concerning $G \subset GL(V)$ is valid for arbitrary Lie group $G$! ■
3.5 Characteristic classes

The last theorem gives that we can extract the Chern-Weil homomorphism of $\Phi$ via any local direct connection $\tau$ on the level of differential forms. The Chern-Weil homomorphism of $\Phi$ is really the Chern-Weil homomorphism of the Lie algebroid $A(\Phi)$ of $\Phi$.

We recall the construction of the Chern-Weil homomorphism for Lie algebroids

Consider a transitive Lie algebroid $A$ with the Atiyah sequence

$$0 \to g \to A \to TM \to 0$$

with the adjoint bundle of Lie algebras $g$.

The Chern-Weil homomorphism for transitive Lie algebroid $A$ is defined as follows:

$$h_A : \bigoplus_{k \geq 0} \left( \text{Sec} \bigwedge^k g^* \right)_{0} \to \mathcal{H}_{dR}(M)$$

$$\Gamma \mapsto \left[ \frac{1}{k!} \langle \Gamma, \Omega \vee \ldots \vee \Omega \rangle \right]$$

where $\Omega \in \Omega^2_E(M; g)$ is the curvature tensor of any connection in $A$, whereas $\left( \text{Sec} \bigwedge^k g^* \right)_{0}$ is the space of invariant cross-sections of $\bigwedge^k g^*$ with respect to the adjoint representation of $A$ on $\bigwedge^k g^*$, i.e. $\Gamma \in \left( \text{Sec} \bigwedge^k g^* \right)_{0}$ if and only if

$$\forall \xi \in \text{Sec}_A \forall \sigma_1, \ldots, \sigma_k \in \text{Sec}_g \left( \langle \gamma \circ \xi \rangle (\Gamma, \sigma_1 \vee \ldots \vee \sigma_k) = \sum_{i=1}^{k} \langle \Gamma, \sigma_1 \vee \ldots \vee [\xi, \sigma_i] \vee \ldots \vee \sigma_k \rangle \right)$$

The nontriviality of $h_A$ means, of course, that in $A$ there is no flat connection.
We explain also that $\Omega \wedge \ldots \wedge \Omega$ is the usual skew multiplication of differential forms with values multiplying symmetrically

$$\Omega \wedge \ldots \wedge \Omega \left( x; v_1, \ldots, v_{2k} \right) = \sum_{\sigma \in \Sigma^{2k}} \text{sgn} \; \sigma \cdot \Omega \left( x; v_{\sigma_1}, v_{\sigma_2} \right) \wedge \ldots \wedge \Omega \left( x; v_{\sigma_{2k-1}}, v_{g_{\sigma_{2k}}} \right) \in \bigwedge^k g_x.$$  

For example: for the Lie algebroid $A(P)$ of a principal fibre bundle $P$ ($P$ is assumed to be connected) and equivalently for Lie algebroid of the Ehresmann Lie groupoid $\Phi = PP^{-1}$, there is a natural isomorphism of algebras $\nu$ such that the diagram commutes

$$\bigoplus_{k \geq 0} \left( \text{Sec} \bigwedge^k g^* \right)_I^{h_{A(P)}} \cong \nu \bigwedge^k_{h_{P}} \left( \text{H}_{dR} (M) \right)$$

which means that the Chern-Weil homomorphism of a Lie algebroid is some generalization of this notion known on the ground of principal bundles. On the other hand, this also means that the Chern-Weil homomorphism of a principal bundle is a characteristic feature of its Lie algebroid (for connected principal bundles).
In addition, we must point out two things:

1) A Lie algebroid is - in some sense - a simpler structure than a principal bundle. Namely, nonisomorphic principal bundles can possess isomorphic Lie algebroids. For example, there exists a nontrivial principal bundle for which the Lie algebroid is trivial (the nontrivial Spin(3)-structure of the trivial principal bundle $\mathbb{R}P(5) \times SO(3)$).

2) There exist other sources of Lie algebroids than principal bundles, for example, transversally complete foliations, nonclosed Lie subgroups, Poisson manifolds and other.
**Example 19** Let $\Phi = GL(E)$ be a Lie groupoid of all linear fibre isomorphisms. Then the Atiyah sequence of $A(\Phi)$ is

$$0 \to \text{End}(E) \to A(\Phi) \to TM \to 0.$$

Consider the Chern-Weil homomorphism

$$h : \bigoplus_{k \geq 0} \left( \text{Sec} \bigwedge^k \text{End}(E)^* \right) \to H_{dR}(M), \Gamma \mapsto \left[ \frac{1}{k!} \langle \Gamma, \Omega \wedge \ldots \wedge \Omega \rangle \right]$$

where $\Omega \in \Omega^2(M; \text{End}(E))$ is the curvature tensor of any connection in $\Phi$. 

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**Pontryagin classes.** Take the invariant cross section $C_k \in \Gamma \left( \text{Sec} \, \bigwedge^k \text{End} \, (E)^* \right)$ by

$$C_{k|x} = \text{tr} \left( \varphi_1 \Box \ldots \Box \varphi_k \right)$$

where for $\varphi_i \in \text{End} \, (E|_x)$ the linear mapping $\varphi_1 \Box \ldots \Box \varphi_k : \bigwedge^k E|_x \rightarrow \bigwedge^k E|_x$ is defined [Greub-Halperin-Vanstone] by

$$\varphi_1 \Box \ldots \Box \varphi_k (v_1 \wedge \ldots \wedge v_k) = \sum_{\sigma \in \Sigma^k} \text{sgn} \sigma \cdot \varphi_1 (v_{\sigma_1}) \wedge \ldots \wedge \varphi_k (v_{\sigma_k}).$$

Then the Pontryagin class is equal to

$$p_k (E) = p_k (\Phi) = h (C_{2k}) = \frac{1}{(2k)!} \left[ \langle C_{2k}, \bigvee \ldots \bigvee \bigvee \rangle \right]_{\text{2k times}} dR.$$ 

The class $p_k (E)$ is represented by the differential form

$$c \cdot \text{tr} \left( \Omega \Box \ldots \Box \Omega \right).$$

According to the notation of Greub-Halperin-Vanstone, the forms $\bigvee \ldots \bigvee \bigvee \Omega$ and $\Omega \Box \ldots \Box \Omega$ are the usual skew multiplication of differential forms for which the values are multiplicates by the suitable mappings

$$\bigvee : \text{End} \, (E) \times \ldots \times \text{End} \, (E) \rightarrow \bigvee^{2k} \text{End} \, (E).$$

$$\Box : \text{End} \, (E) \times \ldots \times \text{End} \, (E) \rightarrow \text{End} \left( \bigwedge^{2k} E \right).$$

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**Trace classes.** Take the invariant cross section \( \text{Tr}_k \in \Gamma \left( \text{Sec}^k \text{End} \left( E \right)^* \right) \) by
\[
\text{Tr}_k (\varphi_1, \ldots, \varphi_k) = \sum_{\sigma \in \Sigma_k} \text{tr} \left( \varphi_{\sigma_1} \circ \cdots \circ \varphi_{\sigma_k} \right).
\]
Then the trace class is equal to
\[
\text{tr}_k (E) = \text{tr}_k (\Phi) = h(\text{Tr}_2) = \frac{1}{(2k)!} \left[ \langle \text{Tr}_{2k}, \underbrace{\Omega \lor \cdots \lor \Omega}_{2k \text{ times}} \rangle \right]_{dR}.
\]
The class \( \text{tr}_k (E) \) is represented by the differential form
\[
c \cdot \text{tr} (\Omega \circ \cdots \circ \Omega)
\]
(the values of the skew multiplication of \( \Omega \circ \cdots \circ \Omega \) are multiplied by the composing of the linear mapping.)
Pfaffian class for oriented $2k$-dimensional vector bundle $E$. Take the invariant cross-section $pf \in \Gamma \left( \text{Sec} \sqrt[k]{Sk(E)} \right)$

$$pf^F (\varphi_1, ..., \varphi_k) = \langle e, \beta^{-1} (\varphi_1) \wedge ... \wedge \beta^{-1} (\varphi_k) \rangle$$

where $\beta : \bigwedge^2 (F) \xrightarrow{\cong} Sk_F$, $\beta (x \wedge y) (z) = \langle x, z \rangle y - \langle y, z \rangle x$ and $e \in \bigwedge^k F$ determine the orientation and $|\langle e, e \rangle| = 1$. Then the Atiyah sequence of $A(Iso E)$ is

$$0 \rightarrow Sk (E) \rightarrow A(Iso E) \rightarrow TM \rightarrow 0.$$ 

and the Pfaffian class is equal to $i^k \cdot h (pf)$ and it is represented by

$$c \cdot \langle \Delta, (\beta^{-1} \Omega) \wedge ... \wedge (\beta^{-1} \Omega) \rangle.$$
Take a local direct connection $\tau$ in $\Phi$ and consider once again the curvature form $\Omega(\Psi^\tau_2) \in \Omega^2(M; g)$,

$$\Omega^\tau_{x_0} = 2 \sum_{i<j} \left( \frac{\partial^2 \hat{\tau}}{\partial x^i \partial x^{m+i}} - \frac{\partial^2 \hat{\tau}}{\partial x^j \partial x^{m+j}} + \left[ \frac{\partial \hat{\tau}}{\partial x^i}, \frac{\partial \hat{\tau}}{\partial x^j} \right]^R \right)_{|_{(x_0, x_0)}} dx^i \wedge dx^j.$$  

We known that $\tau$ induces a usual connection $\nabla^\tau$ in $A(\Phi)$ and that the curvature of it is a constant time the form $\Omega(\Psi^\tau_2)$,

$$\Omega^\tau = 4 \cdot \Omega(\Psi^\tau_2).$$

In conclusion, the Chern-Weil homomorphism of $\Phi$ (i.e. of the $A(\Phi)$) can be extracted via $\tau$ on the level of differential forms by

$$(\Gamma, \Omega \vee ... \vee \Omega) = 4^k (\Gamma, \Omega(\Psi^\tau_2) \vee ... \vee \Omega(\Psi^\tau_2)).$$

**Problem 20** How can we express the form

$$\Omega(\Psi^\tau_2) \vee ... \vee \Omega(\Psi^\tau_2) \in \Omega^{2k}(M; \sqrt{g})$$

with the help of $\Omega(\Psi^{2k}_2)$? and the form $(\Gamma, \Omega(\Psi^\tau_2) \vee ... \vee \Omega(\Psi^\tau_2))$ for an invariant cross-section $\Gamma \in \left( \text{Sec} \sqrt{k} g^* \right)_I$ with the help $\Gamma$ and $\Omega(\Psi^{2k}_2)$?
We know that for $\Phi = GL(E)$ we have the adjoint Lie algebra bundle $g$ is equal to the vector bundle of linear homomorphisms $Aut(E)$. Therefore the Atiyah sequence of $A(\Phi)$ equals

$$0 \rightarrow Aut(E) \rightarrow A(\Phi) \rightarrow TM \rightarrow 0.$$ 

Using the composition of linear homomorphisms

$$\circ: Aut(E) \times \ldots \times Aut(E) \rightarrow Aut(E)$$

we can obtain the following theorem.

**Theorem 21** The equality holds

$$\Omega(\Psi^r_{2k}) = c \cdot \Omega(\Psi^r_2) \circ \ldots \circ \Omega(\Psi^r_2) \ (k \ times)$$

or equivalently

$$\Omega(\Psi^r_{2k}) = c_1 \cdot \Omega^r \circ \ldots \circ \Omega^r \ (k \ times).$$

THE END
Remark concerning the above Problem (20): For an arbitrary Lie groupoid $\Phi$ we consider

$$\Psi_{2k}^r (x_0, x_1, \ldots, x_{2k}) = \tau (x_0, x_1) \cdot \tau (x_1, x_2) \cdot \ldots \cdot \tau (x_{2k-1}, x_{2k}) \cdot \tau (x_{2k}, x_0) \in \Phi_{x_0}^r.$$  

We define additionally

$$\tilde{\Psi}_{2k}^r (x_0, x_1, \ldots, x_{2k})$$

$$= \tau (x_0, x_1) \cdot \tau (x_1, x_2) \cdot \tau (x_2, x_3) \cdot \tau (x_3, x_4) \cdot \ldots \cdot \tau (x_{2k-2}, x_{2k-1}) \cdot \tau (x_{2k-1}, x_{2k}) \cdot \tau (x_{2k}, x_0)$$

$$= \tau (x_0, x_1) \cdot \tau (x_1, x_2) \cdot \tau (x_2, x_3) \cdot \tau (x_3, x_4) \cdot \tau (x_4, x_0) \cdot \ldots \cdot \tau (x_0, x_{2k-1}) \cdot \tau (x_{2k-1}, x_{2k}) \cdot \tau (x_{2k}, x_0)$$

i.e.

$$\tilde{\Psi}_{2k}^r (x_0, x_1, \ldots, x_{2k}) = \Psi_2^r (x_0, x_1, x_2) \cdot \Psi_2^r (x_0, x_3, x_4) \cdot \ldots \cdot \Psi_2^r (x_0, x_{2k-1}, x_{2k})$$

Clearly $\Psi_{2k}^r \neq \tilde{\Psi}_{2k}^r$ in general, but (I think) the equality holds infinitesimally

$$\Omega (\Psi_{2k}^r) (x) = \Omega (\tilde{\Psi}_{2k}^r) (x).$$  

(3)

Next, using (3), we can try to calculate $\Omega (\Psi_{2k}^r) (x)$ via $\Omega (\Psi_2^r) (x)$.