A LOCAL PROPERTY OF THE SUBSPACES OF EUCLIDEAN SPACES

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1. Preliminares

Let $M \neq \emptyset$ be a set and $C$ an arbitrary set of real functions defined on $M$. We denote by $\tau_C$ the weakest topology on $M$ such that all functions belonging to $C$ are continuous. For any set $A$ contained in $M$ we denote by $\tau_C|A$ the set of functions of the form $\alpha|A$ where $\alpha \in C$. We denote by $C|A$ the set of all real functions on $A$ such that for any point $p$ of $A$ there exists in $\tau_C$ an open neighbourhood $U$ of $p$ and a function $\alpha \in C$, such that $\beta|A \cap U = \alpha|A \cap U$. It is easy to verify that, for any set $A \subset M$, we have $\tau_C|A = \tau_C|A$. In particular $\tau_{C_M} = \tau_C$. We denote by $scC$ the set of all real functions of the form $\omega(\alpha_1,\ldots,\alpha_n)$, where $\omega \in \mathcal{E}_n$, $\alpha_1,\ldots,\alpha_n \in C$ and $n$ belongs to the set of all positive integers $\mathbb{N}$ and $\mathcal{E}_n$ is the set of all real $C^\infty$-functions on $n$-dimensional Euclidean space $E^n$. An ordered pair $(M,C)$ such that $C_M = C = scC$ is called to be a differential space. The set $C$ is called the differential structure of this differential space [1], [2], [6].

For a set $C$ of real functions defined on $M$, the set $(scC)_M$ is the smallest differential structure on $M$ including the set $C$. $(M,(scC)_M)$ is called the differential space generated by $C$.

If $(M,C)$ is a differential space and $A \subset M$, then $(A,C_A)$ is also a differential space called the differential subspace of $(M,C)$ [1]. It is easy to see that $C_A = (C|A)_A$.

By a vector tangent to a differential space $(M,C)$ at a point $p$ of $M$ we mean any linear mapping $v : C \to E$ which fulfils Leibniz’s condition at the point $p$:

$$v(\alpha \beta) = v(\alpha) \beta(p) + \alpha(p) v(\beta) \quad \text{for all} \quad \alpha, \beta \in C.$$ 

We shall denote by $(M,C)_p$ or $M_p$ a linear space of all vectors tangent to $(M,C)$ at the point $p \in M$.

Any real $C^\infty$-manifold $M$ will be identified with the differential space $(M,C^\infty(M))$, where $C^\infty(M)$ is the set of all smooth real functions.

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on $M$. In particular, we denote $C^\infty (E^n)$ by $\mathcal{E}_n$ and we call the pair $(E^n, \mathcal{E}_n)$ the $n$-dimensional Euclidean differential space.

It is easy to verify that for $\emptyset \neq M \subset E^n$, $n \in \mathbb{N}$,

$$\mathcal{E}_{nM} = \left( sc \left\{ \pi^i|_{M}; i = 1, \ldots, n \right\} \right)_M$$

where $\pi^i ((x^1, x^2, \ldots, x^n)) = x^i$ for any $(x^1, x^2, \ldots, x^n) \in E^n$. The topological space $(M, \tau_{\mathcal{E}_{nM}})$ is a subspace of the topological space $(E^n, \tau_{\mathcal{E}_n})$.

In the sequel the symbol $\tau_M$ will be used instead of $\tau_{\mathcal{E}_{nM}}$. Using a partition of unity it may be proved that $E^n_M$ is the set of all functions of the form $\alpha |_{M}$, where $\alpha$ is a $C^\infty$-function on an open set $U$ in $E^n$ including $M$. The basic result of this paper consists in the following theorem.

**Theorem 1.** For any $p \in M \subset E^n$ the integer $m = \dim (M, E^n_M)_p$ is the smallest one that there exists in $\tau_M$ an open neighbourhood $U$ of the point $p$ which is included in an $m$-dimensional $C^\infty$-surface of $E^n$.

2. THE PROOF OF BASIC RESULT

From now on we fix the integer $k > 0$, and the non empty set $M \subset E^k$. For brevity we write $\mathcal{E} := \mathcal{E}_k, C := \mathcal{E}_M, M_p := (M, C)_p, E^k_p := (E^k, \mathcal{E}_k)_p$.

The mappings $L_1 : M_p \rightarrow E^k_p$ and $L_2 : E^k_p \rightarrow E^k$ defined by

$$(1) \quad (L_1 (v)) (f) := v (f|_{M}) \quad \text{for} \quad v \in K_p \quad \text{and} \quad f \in \mathcal{E},$$

$$(2) \quad L_2 (\bar{v}) := \left( \bar{v} (\pi^1), \ldots, \bar{v} (\pi^k) \right) \quad \text{for} \quad \bar{v} \in E^k_p,$$

are respectively a linear monomorphism and a linear isomorphism of suitable linear spaces (c.f. [1]).

Let $\partial_i f (p)$ denote $i$-th partial derivative of the function $f \in \mathcal{E}$ at the point $p \in E^k, i = 1, \ldots, k$. If we denote $f_{\hbar} (p) := h^i \partial_i f (p)$ where $h = (h^1, \ldots, h^k) \in E^k$, we have

$$(3) \quad \bar{v} (f) = \partial_i f (p) \bar{v} (\pi^i) = f_{\bar{v}(\pi^i)} (p) \quad \text{for} \quad \bar{v} \in E^k_p,$$

(the sumation convention is used here). Let $L := L_2 \circ L_1 : M_p \rightarrow E^k$ and

$$(4) \quad \bar{M}_p = \left\{ L (\bar{v}) \in E^k; \bar{v} \in M_p \right\}.$$

It is easy to see that the mapping $L : M_p \rightarrow \bar{M}_p$ makes these linear spaces isomorphic to each other. We have

$$(5) \quad \left\{ \begin{array}{ll} L (v) = (v (\pi^1|_{M}), \ldots, v (\pi^k|_{M})) & \text{for} \quad v \in M_p, \\
 v (f|_{M}) = f_{L (v)} & \text{for} \quad v \in M \quad \text{and} \quad f \in \mathcal{E}_k. \end{array} \right.$$
Lemma 1. For \( p \in M, \ h \in E^k, \ k \in \mathcal{N} \) the following properties are equivalent:

(a) \( h \in \bar{M}_p \),

(b) there exists a mapping \( \bar{v} : \mathcal{E}|M \rightarrow E \) such that

\[
\bar{v} (f|M) = f|_h (p) \quad \text{for} \quad f \in \mathcal{E}.
\]

Proof. The implication (a) \( \Rightarrow \) (b) follows immediately from (4) and (5) by putting \( \bar{v} := v|_{(E|M)}, v \in M_p \) and \( h = L(v) \).

In order to prove the implication (b) \( \Rightarrow \) (a) let us suppose that \( h \) fulfills (b) and consider the set of functions \( E|M \). From (b) it follows that \( \bar{v} \) is the linear mapping of \( E|M \) into \( E \) fulfilling the Leibniz’s condition at the point \( p \):

\[
\bar{v} (\alpha\beta) = \bar{v} (\alpha) \beta (p) + \alpha (p) \bar{v} (\beta) \quad \text{for} \quad \alpha, \beta \in \mathcal{E}|M.
\]

By using this conditions and linearity of \( \bar{v} \) one can easy verify that \( \bar{v} (\alpha) = 0 \) for each function \( \alpha \in E|M \) equal to 0 at an open neighbourhood of the point \( p \). As a consequence of this the mapping \( v : C \rightarrow E \) defined by

\[
v (\alpha) := \bar{v} (f|M) \quad \text{for} \quad \alpha \in C,
\]

where \( f \in \mathcal{E} \) is a function such that \( f|U = \alpha|U \) for some set \( U \in \tau_M \) including the point \( p \), is well defined. The function \( v \) is linear and fulfils Leibniz’s condition so it belongs to \( M_p \). For \( i = 1, \ldots, k \) we have

\[
v (\pi_i|M) = \pi_i|_h (p) = \bar{v} (\pi_i|M) = h_i, \quad \text{where} \quad h = (h^1, \ldots, h^k), \quad \text{so from} \ (5) \ \text{we have:} \quad L(v) = (h^1, h^2, \ldots, h^k) = h. \quad \text{The Lemma is proved.} \]

Lemma 2. For \( h \in E^k \) and \( p \in M \) the following conditions are equivalent:

(a) \( h \in \bar{M}_p \),

(b) \( f|_h (p) = 0 \) for any \( f \in \mathcal{E} \) equal to 0 on \( M \).

Proof. It is easy to see that the conditions (b) in Lemmas 1 and 2 are equivalent to each other. \( \square \)

For any \( f \in \mathcal{E} \) and \( p \in E^k \) we denote grad \( f (p) := (\partial_i f (p), \ldots, \partial_k f (p)) \).

Lemma 3. Let \( p = (0, \ldots, 0) \in M \subset E^k \) and \( e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \) (1 in the \( i \)-th position), \( 1 \leq i \leq k \). If \( m := \dim M_p, \ 1 \leq m \leq k−1 \) and \( e_1, \ldots, e_m \in \bar{M}_p \) then there exists functions \( f^{m+1}, \ldots, f^k \in \mathcal{E} \) equal to 0 on \( M \) and such that \( \partial_if^j (p) = \delta^j_i \), where \( \delta^j_i = 1 \) for \( i = j \) and \( \delta^j_i = 0 \) for \( i \neq j \).

Proof. Let the assumptions of the Lemma be satisfied. Then

(6) \[ \bar{M}_p = \text{Lin} (e_1, \ldots, e_m), \]
where $\text{Lin} \left( e_1, ..., e_m \right)$ is the linear subspace of $E^k$ spanned by $e_1, ..., e_m$. We put $K := \{ h \in E^k; h = \text{grad }f(p), \ f \in E \text{ and } f = 0 \text{ on } M \}$. $K$ is a linear subspace of $E^k$ and $e_i \perp K$ with respect to the canonical scalar product in $E^k$ so $K \subset \text{Lin} \left( e_{m+1}, ..., e_k \right)$ (see Lemma 2). We shall prove more, namely that $K = \text{Lin} \left( e_{m+1}, ..., e_k \right)$. If the above equality is not satisfied, then there exists a non-zero vector $h \in \text{Lin} \left( e_{m+1}, ..., e_k \right)$, such that $K \perp h$. Hence $f(h) = \text{grad }f(p) \cdot h = 0$ for $f = 0$ on $M$, and $h \in M_p$ (Lemma 2), but this contradicts (6). From above equality we obtain the existence of functions $f_{m+1}, ..., f_k \in E_k$ equal to zero on $M$, such that $\text{grad }f_j(p) = e_j$ or equivalently $\partial_i f_j(p) = \delta^j_i$. The Lemma is proved.

**Proposition 1.** Let $p \in M \subset E^k$. If $0 < m := \dim M_p \leq k$ then there exist non empty sets: $U$ open in $\tau_M$ and $V$ open in $\tau_{E_m}$, and regular 1-1 $C^\infty$-mapping $\phi : V \rightarrow E^k$ such that

$$p \in U \subset \{ \phi(u) \in E^k; u \in V \}.$$  

**Proof.** If $m = k$, the proposition evidently holds. We suppose that $1 \leq m < k$. We can assume, without loss of generality, that $p = (0, ..., 0) \in E^k$ and $M_p = \text{Lin} \left( e_1, ..., e_m \right)$. We denote $q = (x^1, ..., x^k) = (u, w)$ where $u = (x^1, ..., x^m)$ and $w = (x^{m+1}, ..., x^k)$. Let $f_j, j = m + 1, ..., k$, are functions as in Lemma 3. We define a mapping $F : E^k \rightarrow E^{k-m}$ by

$$F(q) := (f^{m+1}(q), ..., f^k(q)) \text{ for } q \in E^k.$$  

This mapping has the following properties:

(a) $F(q) = F(u, w) = 0$ for $q = (u, w) \in M$,
(b) $F$ is $C^\infty$-mapping,
(c) $F$ is regular at the point $p = (\bar{u}, \bar{w})$.

From the inverse mapping theorem it follows that there exists:

(d) a set $U' \in \tau_{E^k}$ such that $p \in U'$,
(e) a set $V \in \tau_{\mathbb{R}^m}$ such that $\bar{u} \in V$,
(f) a $C^\infty$-mapping $\psi : V \rightarrow E^{k-m}$ such that for any $u \in V$ we have $F(u, \psi(u)) = 0$,

(g) if $F(q) = 0$ and $q = (u, w) \in U'$ then $u \in V$ and $w = \psi(u)$.

It is evident that $U := U' \cap M, V$ and $\phi(u) := (u, \psi(u))$ for $u \in V$ fulfill conditions of Proposition 1.

□

Now, we examine the case of $\dim M_p = 0$, which was not considered above.

**Proposition 2.** Let $p \in M \subset E^k$. If $\dim M_p = 0$ then the point $p$ is isolated in $M$. 
Proof. Let us set \( |q| := \sqrt{(x^1)^2 + \cdots + (x^k)^2} \) for any \( q = (x^1, \ldots, x^k) \in E^k \).

Let us assume the point \( p \) is not isolated. Then there exists a sequence \((p_n)\) of points of \( M \) different from \( p \) and convergent to \( p \). For the sequence \( h_n = \frac{p_n - p}{|p_n - p|}, n \in \N \) of points of \( S^{k-1} \) we can find a subsequence \( h_{n_i} \) convergent to a point \( h \in S^{k-1} \). One can easy see that for any \( f \in E \)
\[
\lim_{i \to \infty} \frac{f(p_{n_i}) - f(p)}{|p_{n_i} - p|} = f_h(p).
\]

It easy to see that left side of this sequence defines mapping \( \bar{v} : E| M \to E \) such that \( \bar{v}(f| M) = f_h(p) \). From Lemma 1 \( h \in M_p \) so \( \dim M_p \neq 0 \), which ends of the proof.

Theorem 1 results easily from Propositions 1 and 2. In that Theorem an non-empty discrete subset of \( E^n \) is called a 0-dimensional \( C^\infty \)-surface in \( E^n \).

3. Corollaries

We say, that differential space \((N,D)\) can be diffeomorphically em-beded into the differential space \((L,H)\) if there exists a subset \( L' \subset L \) such that \((L',H_{L'})\) and \((N,D)\) are diffeomorphic to each other. In the sequel we shall consider only differential spaces \((N,D)\) such that any point \( p \in N \) has a neighbourhood \( V \) such that \((V,D_V)\) can be em-beded into \((E^n(p),E_{n(p)})\) for some \( n(p) \in \N \). From Theorem 1 we obtain:

**Corollary 1.** For a point \( p \) of the differential space \((N,D)\) there exist a set \( V \in \tau_D \) and an \( n \)-dimensional \( C^\infty \)-manifold \( \tilde{N}, C^\infty \left( \tilde{N} \right) \), \( n := \dim N_p \), such that \( p \in V \subset \tilde{N} \) and \( D_V = C^\infty \left( \tilde{N} \right)_V \). The inequality
\[
\dim M_q \leq \dim M_p
\]
is fulfilled for any point \( q \in V \).

**Corollary 2.** If \((N,D)\) is a differential space such that \((N,\tau_D)\) is separable and if there exists \( n \in \N \) such that for any \( p \in N \) \( \dim (N,D)_p \leq n \), then topological dimension of \((N,\tau_D)\) does not exceed \( n \).

*Proof.* This results easily from Corollary 1. □

Differential spaces which have tangent spaces of constant dimension are the most interesting. For a differential space \((N,D)\) and \( i = 0, 1, \ldots \) we shall denote by \( N^i \) union of all sets \( V \in \tau_D \) such that \( \dim (N,D) = i \)
for any \( q \in V \). If \( N^i \) is not empty then \( (N^i, D_{N^i}) \) is a differential subspace of \((N, D)\) and for any \( q \in N^i \) \( \dim (N^i, D_{N^i}) = i \). From Corollary 1 we obtain

**Corollary 3.** For any differential space \((N, D)\) the set \( \bigcup_{i=0}^\infty N^i \) is open and dense in the topological space \((N, \tau_D)\).

*Proof.* For any subset \( A \subset N \) we denote its closure in \((N, \tau_D)\) by \( \bar{A} \). We shall use mathematical induction. Let \( p \in N \). It is easy to see, that \( \dim (N, D)_p \geq 0 \). If \( \dim (N, D)_p = 0 \) then the point \( p \) is isolated in \((N, \tau_D)\) and \( p \in N^0 \), see Corollary 1, \( p \in \bigcup_{i=0}^\infty N^i \). Suppose, that for any \( q \in N \) such that \( 0 \leq \dim (N, D)_q \leq m - 1 \) we have \( q \in \bigcup_{i=0}^\infty N^i \).

For any point \( p \in N \) such that \( \dim (N, D)_p = m \) there exists an open neighbourhood \( V \) of \( p \) such that \( \dim (N, D)_q \leq m \) for any point \( q \in V \) (Corollary 1). Let \( U \in \tau_D \) be a set containing the point \( p \). If for any \( q \in U \cap V \) \( \dim (N, D)_q = m \), then \( p \in N^m \) and \( p \in \bigcup_{i=0}^\infty N^i \). If it is not true then there exists a point \( q \in U \cap V \), such that \( \dim (N, D)_q \leq m - 1 \). From the induction hypothesis, the point \( q \in \bigcup_{i=0}^\infty N^i \), so \( U \cap \bigcup_{i=0}^\infty N^i \neq \emptyset \). This is true for any set \( U \in \tau_D \) containing the point \( p \), so we have \( p \in \bigcup_{i=0}^\infty N^i \). The corollary is proved. \( \square \)

By virtue of Corollary 1 any point \( p \) of differential space \((N, D)\) such that \( \dim (N, D)_p = k \) has a neighbourhood \( V \) such that \((V, D_V)\) can be diffeomorphically embedded in \((E^k, \mathcal{E}_k)\). Hence it is interesting to consider the differential subspace \((M, \mathcal{E}_{kM})\) of \((E^k, \mathcal{E}_k)\) for which there exists a point \( p \in M \) such that \( \dim (M, C)_p = k \).

**Corollary 4.** Let \( p \in M \subset E^k \). \( \dim (M, \mathcal{E}_{kM})_p = k \) if and only if for any \( f \in \mathcal{E}_k \) equal to 0 on \( M \) \( \partial_i f (p) = 0 \) for \( i = 1, 2, ..., m \).

*Proof.* We get this immediately from Lemma 2, as \( e_i \in M_p, i = 1, ..., k \). \( \square \)

**Corollary 5.** Let \( \emptyset \neq M \subset E^k \). Then \( \dim (M, \mathcal{E}_{kM})_p = k \) for any \( p \in M \) if and only if for any \( f \in \mathcal{E}_k \) equal to 0 on \( M \) all partial derivatives of any order are equal to 0 on \( M \).

*Proof.* This corollary follows easily, by induction, from Corollary 4. \( \square \)

By virtue of above Corollary, any subset \( M \subset E^k \) such that \((M, \mathcal{E}_{kM})\) has the constant dimension \( k \), has the same property, as any open set of \( E^k \): the value of the partial derivatives of a function \( f \in \mathcal{E}_k \) at a point \( p \) are uniquely determined by the values of the function on \( M \). For a differential space \((N, D)\) a linear mapping \( X : D \to D \) such
that $X(\alpha \beta) = X(\alpha) \beta + \alpha X(\beta)$ is called a vector field on $(N, D)$ [1]. It is easy to see that for any point $p \in N$ the function $X_p: D \to E$ defined by $X_p(\alpha) := (X\alpha)(p)$ for $\alpha \in D$ is a vector belonging to $(N, D)_p$.

Corollary 6. Let $(N, D)$ be a differential space. A point $p$ belongs to $\bigcup_{i=0}^{\infty} N^i$ if and only if there exists vector fields $X_1, \ldots, X_s$ on $(N, D)$ such that $\{X_{1p}, \ldots, X_{sp}\}$ is the basis of $(N, D)_p$.

Proof. If $X_1, \ldots, X_k$ are such vector fields on $(N, D)$ that $X_{1p}, \ldots, X_{kp}$ is a basis of $(N, D)_p$ then there exists a set $V \in \tau_D$ such that $p \in V'$ and $X_{1q}, \ldots, X_{kq}$ are linear independent for any $q \in V'$ (cf. [1]). As there exists an open neighbourhood $V''$ of $p$ such that for any $q \in V''$ $\dim (N, D)_q = k$ (Corollary 1), for any $q \in V' \cap V''$ $\dim (N, D)_q = k$ and $p \in N^k \subset \bigcup_{i=0}^{\infty} N^i$.

Now we shall prove the other implication. For the point $p \in N^0$ the proof is trivial. Let $p \in N^k$, $k > 0$ and $U$ be such an open neighbourhood of the point $p$ that $(U, D_U)$ is diffeomorphic to $(V, E_{kV})$ for certain $V \subset E^k$ and $\dim (V, E_{kV})_q = k$ for any $q \in V$. It is sufficient to prove Corollary for $(V, E_{kV})$.

For $q \in V$ and $\alpha \in E_{kV}$ there exists an open neighbourhood $V_q$ of $q$ and a function $f_{\alpha,q} \in E$ such that $\alpha|V_q = f_{\alpha,q}|V_q$. By virtue of Corollary 5 the functions $X_i: E_{kV} \to E_{kV}$, $i = 1, 2, \ldots, k$, defined for $\alpha \in E_{kV}$, by

$$(X_i\alpha)(q) = \partial_i (f_{\alpha,q})(q) \quad \text{for} \quad q \in V$$

are well defined. It can be easily verified that they are vector fields on $(V, E_{kV})$ and $X_{1q}, \ldots, X_{kq}$ is the basis of $(V, E_{kV})_q$ for any $q \in V$. \hfill \Box

4. Examples

**Example 1.** Let $M \subset E^k$ be dense in $E^k$. Then by Corollary 1 the dimension of $(M, E_{kM})_p$ is $k$ for any $p \in M$.

**Example 2.** The graph of the function $f: E \to E$ which is $x^2$ for $x > 0$ and $0$ for $x \leq 0$ has the tangent space of dimension 1 at all points except for the point $(0, 0)$, where it has tangent space of dimension 2. It results easily from Corollary 1.

**Example 3.** The graph of the function $g: E \to E$ of class $C^1$ which is not of class $C^\infty$ at any point is a differential subspace of $(E^2, E_2)$ of constant dimension 2. It results easily from Corollary 1.

**Example 4.** Let $M \subset E^k$. If topological dimension of any non empty open subset of $M$ is $k$ then $\dim (M, E_{kM})_p = k$ for any $p \in M$. This follows easily from Corollary 2.
References


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