Fuzzy version of the Huthinson–Barnsley theory of fractals

Filip Strobin

(with Elismar Oliveira)

Institute of Mathematics,
Lodz University of Technology,
Łódź, Poland

basics of the Hutchinson–Barnsley theory

Assume that X is a metric space.

(*) By $\mathcal{K}(X)$ we denote the family of all nomepty and compact subsets of X.

(*) A finite family of continuous selfmaps of X will be called an iterated function system (IFS).

(*) If $\mathcal{F} = \{f_1, ..., f_n\}$ is an IFS, then we define $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$\mathcal{F}(K) := \bigcup_{i=1}^{n} f_i(K)$$

Theorem(Hutchinson, Barnsley, 1980s')
If X is complete and $\mathcal{F}$ is an IFS on X consisting of Banach (or weak) contractions, then there exists the unique $A_{\mathcal{F}} \in \mathcal{K}(X)$ such that

$$A_{\mathcal{F}} = \mathcal{F}(A_{\mathcal{F}}) = \bigcup_{i=1}^{n} f_i(A_{\mathcal{F}}).$$

Moreover, for every $K \in \mathcal{K}(X)$, the sequence of iterates $\mathcal{F}^{(k)}(K)$ converges to $A_{\mathcal{F}}$ w.r.t. the Hausdorff metric.

basics of the Hutchinson–Barnsley theory

Assume that X is a metric space.

(*) By $\mathcal{K}(X)$ we denote the family of all nomepty and compact subsets of X.

(*) A finite family of continuous selfmaps of X will be called an iterated function system (IFS).

(*) If $\mathcal{F} = \{f_1, ..., f_n\}$ is an IFS, then we define $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$\mathcal{F}(K) := \bigcup_{i=1}^{n} f_i(K)$$

Theorem(Hutchinson, Barnsley, 1980s')
If X is complete and $\mathcal{F}$ is an IFS on X consisting of Banach (or weak) contractions, then there exists the unique $A_\mathcal{F} \in \mathcal{K}(X)$ such that

$$A_\mathcal{F} = \mathcal{F}(A_\mathcal{F}) = \bigcup_{i=1}^{n} f_i(A_\mathcal{F}).$$

Moreover, for every $K \in \mathcal{K}(X)$, the sequence of iterates $\mathcal{F}^{(k)}(K)$ converges to $A_\mathcal{F}$ w.r.t. the Hausdorff metric.

basics of the Hutchinson–Barnsley theory

Assume that X is a metric space.

(*) By $\mathcal{K}(X)$ we denote the family of all nomepty and compact subsets of X.

(*) A finite family of continuous selfmaps of X will be called an iterated function system (IFS).

(*) If $\mathcal{F} = \{f_1, ..., f_n\}$ is an IFS, then we define $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$\mathcal{F}(K) := \bigcup_{i=1}^{n} f_i(K)$$

Theorem(Hutchinson, Barnsley, 1980s')
If X is complete and $\mathcal{F}$ is an IFS on X consisting of Banach (or weak) contractions, then there exists the unique $A_{\mathcal{F}} \in \mathcal{K}(X)$ such that

$$A_{\mathcal{F}} = \mathcal{F}(A_{\mathcal{F}}) = \bigcup_{i=1}^{n} f_i(A_{\mathcal{F}}).$$

Moreover, for every $K \in \mathcal{K}(X)$, the sequence of iterates $\mathcal{F}^{(k)}(K)$ converges to $A_{\mathcal{F}}$ w.r.t. the Hausdorff metric.

basics of the Hutchinson–Barnsley theory

Assume that X is a metric space.

(*) By $\mathcal{K}(X)$ we denote the family of all nomepty and compact subsets of X.

(*) A finite family of continuous selfmaps of X will be called an iterated function system (IFS).

(*) If $\mathcal{F} = \{f_1, ..., f_n\}$ is an IFS, then we define $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$\mathcal{F}(K) := \bigcup_{i=1}^{n} f_i(K)$$

Theorem(Hutchinson, Barnsley, 1980s')
If X is complete and $\mathcal{F}$ is an IFS on X consisting of Banach (or weak) contractions, then there exists the unique $A_{\mathcal{F}} \in \mathcal{K}(X)$ such that

$$A_{\mathcal{F}} = \mathcal{F}(A_{\mathcal{F}}) = \bigcup_{i=1}^{n} f_i(A_{\mathcal{F}}).$$

Moreover, for every $K \in \mathcal{K}(X)$, the sequence of iterates $\mathcal{F}^{(k)}(K)$ converges to $A_{\mathcal{F}}$ w.r.t. the Hausdorff metric.

basics of the Hutchinson–Barnsley theory

Assume that X is a metric space.

(*) By $\mathcal{K}(X)$ we denote the family of all nomepty and compact subsets of X.

(*) A finite family of continuous selfmaps of X will be called an iterated function system (IFS).

(*) If $\mathcal{F} = \{f_1, ..., f_n\}$ is an IFS, then we define $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$\mathcal{F}(K) := \bigcup_{i=1}^{n} f_i(K)$$

Theorem(Hutchinson, Barnsley, 1980s')
If X is complete and $\mathcal{F}$ is an IFS on X consisting of Banach (or weak) contractions, then there exists the unique $A_\mathcal{F} \in \mathcal{K}(X)$ such that

$$A_\mathcal{F} = \mathcal{F}(A_\mathcal{F}) = \bigcup_{i=1}^{n} f_i(A_\mathcal{F}).$$

Moreover, for every $K \in \mathcal{K}(X)$, the sequence of iterates $\mathcal{F}^{(k)}(K)$ converges to $A_\mathcal{F}$ w.r.t. the Hausdorff metric.

fuzzy sets

Let X, Y be nonempty sets.

(*) A fuzzy set u on X will be any function $u : X \to [0, 1]$.

(*) If u, v are any fuzzy sets on X, then its union $u \cup v$ is defined by

$$u \cup v(x) := \max\{u(x), v(x)\}, \quad x \in X.$$

(*) A fuzzy set u on X will be called a crisp set, if $u = \chi_A$ for some nonempty $A \subset X$.

(*) If $f : X \to Y$ and $u : X \to [0, 1]$ is a fuzzy set, then we define the fuzzy set $f(u)$ on Y by

$$f(u)(y) := \sup\{u(x) : x \in f^{-1}(y)\},$$

where we additionally assume that $\sup \emptyset = 0$.

Remark

If $A \subset X$ and $f : X \to Y$, then $f(\chi_A) = \chi_{f(A)}$.

fuzzy sets

Let X, Y be nonempty sets.

(*) A fuzzy set u on X will be any function $u : X \to [0, 1]$.

(*) If u, v are any fuzzy sets on X, then its union $u \cup v$ is defined by

$$u \cup v(x) := \max\{u(x), v(x)\}, \quad x \in X.$$

(*) A fuzzy set u on X will be called a crisp set, if $u = \chi_A$ for some nonempty $A \subset X$.

(*) If $f : X \to Y$ and $u : X \to [0, 1]$ is a fuzzy set, then we define the fuzzy set $f(u)$ on Y by

$$f(u)(y) := \sup\{u(x) : x \in f^{-1}(y)\},$$

where we additionally assume that $\sup \emptyset = 0$.

Remark

If $A \subset X$ and $f : X \to Y$, then $f(\chi_A) = \chi_{f(A)}$.

fuzzy sets

Let X, Y be nonempty sets.

(*) A fuzzy set u on X will be any function $u : X \to [0, 1]$.

(*) If u, v are any fuzzy sets on X, then its union $u \cup v$ is defined by

$$u \cup v(x) := \max\{u(x), v(x)\}, \quad x \in X.$$

(*) A fuzzy set u on X will be called a crisp set, if $u = \chi_A$ for some nonempty $A \subset X$.

(*) If $f : X \to Y$ and $u : X \to [0, 1]$ is a fuzzy set, then we define the fuzzy set $f(u)$ on Y by

$$f(u)(y) := \sup\{u(x) : x \in f^{-1}(y)\},$$

where we additionally assume that $\sup \emptyset = 0$.

Remark

If $A \subset X$ and $f : X \to Y$, then $f(\chi_A) = \chi_{f(A)}$.

fuzzy sets

Let X, Y be nonempty sets.

(*) A fuzzy set u on X will be any function $u : X \to [0,1]$.

(*) If u, v are any fuzzy sets on X, then its union $u \cup v$ is defined by

$$u \cup v(x) := \max\{u(x), v(x)\}, \quad x \in X.$$

(*) A fuzzy set u on X will be called a crisp set, if $u = \chi_A$ for some nonempty $A \subset X$.

(*) If $f : X \to Y$ and $u : X \to [0,1]$ is a fuzzy set, then we define the fuzzy set $f(u)$ on Y by

$$f(u)(y) := \sup\{u(x) : x \in f^{-1}(y)\},$$

where we additionally assume that $\sup \emptyset = 0$.

Remark

If $A \subset X$ and $f : X \to Y$, then $f(\chi_A) = \chi_{f(A)}$.

fuzzy sets

Let X, Y be nonempty sets.

(*) A fuzzy set u on X will be any function $u : X \to [0,1]$.

(*) If u, v are any fuzzy sets on X, then its union $u \cup v$ is defined by

$$u \cup v(x) := \max\{u(x), v(x)\}, \quad x \in X.$$

(*) A fuzzy set u on X will be called a crisp set, if $u = \chi_A$ for some nonempty $A \subset X$.

(*) If $f : X \to Y$ and $u : X \to [0,1]$ is a fuzzy set, then we define the fuzzy set $f(u)$ on Y by

$$f(u)(y) := \sup\{u(x) : x \in f^{-1}(y)\},$$

where we additionally assume that $\sup \emptyset = 0$.

Remark

If $A \subset X$ and $f : X \to Y$, then $f(\chi_A) = \chi_{f(A)}$.

fuzzy sets

Let X, Y be nonempty sets.

- (*) A fuzzy set u on X will be any function $u : X \to [0,1]$.
- (*) If u, v are any fuzzy sets on X, then its union $u \cup v$ is defined by

$$u \cup v(x) := \max\{u(x), v(x)\}, \quad x \in X.$$

- (*) A fuzzy set u on X will be called a crisp set, if $u = \chi_A$ for some nonempty $A \subset X$.
- (*) If $f : X \to Y$ and $u : X \to [0,1]$ is a fuzzy set, then we define the fuzzy set $f(u)$ on Y by

$$f(u)(y) := \sup\{u(x) : x \in f^{-1}(y)\},$$

where we additionally assume that $\sup \emptyset = 0$.

Remark

If $A \subset X$ and $f : X \to Y$, then $f(\chi_A) = \chi_{f(A)}$.

fuzzyfication of $\mathcal{K}(X)$

Let X be a metric space and $u : X \to [0, 1]$.

(*) If $\alpha \in [0, 1]$, then its α-cut is defined by

$$[u]^\alpha := \begin{cases} \{x \in X : u(x) \geqslant \alpha\} & \text{if} \quad \alpha > 0 \\ \text{cl}(\{x \in X : u(x) > 0\}) & \text{if} \quad \alpha = 0 \end{cases}.$$

(*) u is called upper semicontiunous (usc) if each set $[u]^\alpha$ is closed.

(*) u is compactly supported, if the set $\text{supp}(u) := [u]^0$ is compact.

(*) normal, if $u(x) = 1$ for some $x \in X$.

fuzzyfication of $\mathcal{K}(X)$

Let X be a metric space and $u : X \to [0,1]$.

(*) If $\alpha \in [0,1]$, then its α-cut is defined by

$$[u]^\alpha := \begin{cases} \{x \in X : u(x) \geqslant \alpha\} & \text{if} \quad \alpha > 0 \\ \mathrm{cl}(\{x \in X : u(x) > 0\}) & \text{if} \quad \alpha = 0 \end{cases}.$$

(*) u is called upper semicontiunous (usc) if each set $[u]^\alpha$ is closed.

(*) u is compactly supported, if the set $\mathrm{supp}(u) := [u]^0$ is compact.

(*) normal, if $u(x) = 1$ for some $x \in X$.

fuzzyfication of $\mathcal{K}(X)$

Let X be a metric space and $u : X \to [0,1]$.

(*) If $\alpha \in [0,1]$, then its α-cut is defined by

$$[u]^\alpha := \begin{cases} \{x \in X : u(x) \geqslant \alpha\} & \text{if} \quad \alpha > 0 \\ \text{cl}(\{x \in X : u(x) > 0\}) & \text{if} \quad \alpha = 0 \end{cases}.$$

(*) u is called upper semicontiunous (usc) if each set $[u]^\alpha$ is closed.

(*) u is compactly supported, if the set $\text{supp}(u) := [u]^0$ is compact.

(*) normal, if $u(x) = 1$ for some $x \in X$.

fuzzyfication of $\mathcal{K}(X)$

Let X be a metric space and $u : X \to [0,1]$.

(*) If $\alpha \in [0,1]$, then its α-cut is defined by

$$[u]^\alpha := \begin{cases} \{x \in X : u(x) \geqslant \alpha\} & \text{if} \quad \alpha > 0 \\ \mathrm{cl}(\{x \in X : u(x) > 0\}) & \text{if} \quad \alpha = 0 \end{cases}.$$

(*) u is called upper semicontiunous (usc) if each set $[u]^\alpha$ is closed.

(*) u is compactly supported, if the set $\mathrm{supp}(u) := [u]^0$ is compact.

(*) normal, if $u(x) = 1$ for some $x \in X$.

fuzzyfication of $\mathcal{K}(X)$

Let X be a metric space and $u : X \to [0,1]$.

(*) If $\alpha \in [0,1]$, then its α-cut is defined by

$$[u]^\alpha := \begin{cases} \{x \in X : u(x) \geqslant \alpha\} & \text{if} \quad \alpha > 0 \\ \mathrm{cl}(\{x \in X : u(x) > 0\}) & \text{if} \quad \alpha = 0 \end{cases}.$$

(*) u is called upper semicontiunous (usc) if each set $[u]^\alpha$ is closed.

(*) u is compactly supported, if the set $\mathrm{supp}(u) := [u]^0$ is compact.

(*) normal, if $u(x) = 1$ for some $x \in X$.

fuzzyfication of $\mathcal{K}(X)$

Assume that X is a metric space. Put

$$\mathcal{K}_F(X) := \{u : X \to [0,1] : u \text{ is usc, compactly supported and normal}\}.$$

For $u, v \in \mathcal{K}_F(X)$, define

$$d_\infty(u,v) := \sup_{\alpha \in [0,1]} h([u]^\alpha, [v]^\alpha),$$

where h is the Hausdorff metric.

Fact

(1) d_∞ is a metric on $\mathcal{K}_F(X)$.

(2) If X us complete [compact], then $\mathcal{K}_F(X)$ is complete [compact].

(3) $A \in \mathcal{K}(X)$ iff $\chi_A \in \mathcal{K}_F(X)$.

(4) For $A, B \in \mathcal{K}(X)$, $d_\infty(\chi_A, \chi_B) = h(A, B)$.

fuzzyfication of $\mathcal{K}(X)$

Assume that X is a metric space. Put

$$\mathcal{K}_F(X) := \{u : X \to [0,1] : u \text{ is usc, compactly supported and normal}\}.$$

For $u, v \in \mathcal{K}_F(X)$, define

$$d_\infty(u,v) := \sup_{\alpha\in[0,1]} h([u]^\alpha, [v]^\alpha),$$

where h is the Hausdorff metric.

Fact

(1) d_∞ is a metric on $\mathcal{K}_F(X)$.

(2) If X us complete [compact], then $\mathcal{K}_F(X)$ is complete [compact].

(3) $A \in \mathcal{K}(X)$ iff $\chi_A \in \mathcal{K}_F(X)$.

(4) For $A, B \in \mathcal{K}(X)$, $d_\infty(\chi_A, \chi_B) = h(A, B)$.

fuzzyfication of $\mathcal{K}(X)$

Assume that X is a metric space. Put

$$\mathcal{K}_F(X) := \{u : X \to [0,1] : u \text{ is usc, compactly supported and normal}\}.$$

For $u, v \in \mathcal{K}_F(X)$, define

$$d_\infty(u,v) := \sup_{\alpha \in [0,1]} h([u]^\alpha, [v]^\alpha),$$

where h is the Hausdorff metric.

Fact

(1) d_∞ is a metric on $\mathcal{K}_F(X)$.

(2) If X us complete [compact], then $\mathcal{K}_F(X)$ is complete [compact].

(3) $A \in \mathcal{K}(X)$ iff $\chi_A \in \mathcal{K}_F(X)$.

(4) For $A, B \in \mathcal{K}(X)$, $d_\infty(\chi_A, \chi_B) = h(A, B)$.

fuzzyfication of $\mathcal{K}(X)$

Assume that X is a metric space. Put

$$\mathcal{K}_F(X) := \{u : X \to [0,1] : u \text{ is usc, compactly supported and normal}\}.$$

For $u, v \in \mathcal{K}_F(X)$, define

$$d_\infty(u,v) := \sup_{\alpha \in [0,1]} h([u]^\alpha, [v]^\alpha),$$

where h is the Hausdorff metric.

Fact

(1) d_∞ is a metric on $\mathcal{K}_F(X)$.

(2) If X us complete [compact], then $\mathcal{K}_F(X)$ is complete [compact].

(3) $A \in \mathcal{K}(X)$ iff $\chi_A \in \mathcal{K}_F(X)$.

(4) For $A, B \in \mathcal{K}(X)$, $d_\infty(\chi_A, \chi_B) = h(A,B)$.

fuzzyfication of $\mathcal{K}(X)$

Assume that X is a metric space. Put

$$\mathcal{K}_F(X) := \{u : X \to [0,1] : u \text{ is usc, compactly supported and normal}\}.$$

For $u, v \in \mathcal{K}_F(X)$, define

$$d_\infty(u,v) := \sup_{\alpha\in[0,1]} h([u]^\alpha, [v]^\alpha),$$

where h is the Hausdorff metric.

Fact

(1) d_∞ is a metric on $\mathcal{K}_F(X)$.

(2) If X us complete [compact], then $\mathcal{K}_F(X)$ is complete [compact].

(3) $A \in \mathcal{K}(X)$ iff $\chi_A \in \mathcal{K}_F(X)$.

(4) For $A, B \in \mathcal{K}(X)$, $d_\infty(\chi_A, \chi_B) = h(A, B)$.

fuzzyfication of $\mathcal{K}(X)$

Assume that X is a metric space. Put

$$\mathcal{K}_F(X) := \{u : X \to [0,1] : u \text{ is usc, compactly supported and normal}\}.$$

For $u, v \in \mathcal{K}_F(X)$, define

$$d_\infty(u,v) := \sup_{\alpha \in [0,1]} h([u]^\alpha, [v]^\alpha),$$

where h is the Hausdorff metric.

Fact

(1) d_∞ is a metric on $\mathcal{K}_F(X)$.

(2) If X us complete [compact], then $\mathcal{K}_F(X)$ is complete [compact].

(3) $A \in \mathcal{K}(X)$ iff $\chi_A \in \mathcal{K}_F(X)$.

(4) For $A, B \in \mathcal{K}(X)$, $d_\infty(\chi_A, \chi_B) = h(A, B)$.

fuzzyfication of $\mathcal{K}(X)$

Assume that X is a metric space. Put

$$\mathcal{K}_F(X) := \{u : X \to [0,1] : u \text{ is usc, compactly supported and normal}\}.$$

For $u, v \in \mathcal{K}_F(X)$, define

$$d_\infty(u,v) := \sup_{\alpha\in[0,1]} h([u]^\alpha, [v]^\alpha),$$

where h is the Hausdorff metric.

Fact

(1) d_∞ is a metric on $\mathcal{K}_F(X)$.

(2) If X us complete [compact], then $\mathcal{K}_F(X)$ is complete [compact].

(3) $A \in \mathcal{K}(X)$ iff $\chi_A \in \mathcal{K}_F(X)$.

(4) For $A, B \in \mathcal{K}(X)$, $d_\infty(\chi_A, \chi_B) = h(A,B)$.

fuzzyfication of $\mathcal{K}(X)$

Assume that X is a metric space. Put

$$\mathcal{K}_F(X) := \{u : X \to [0,1] : u \text{ is usc, compactly supported and normal}\}.$$

For $u, v \in \mathcal{K}_F(X)$, define

$$d_\infty(u,v) := \sup_{\alpha\in[0,1]} h([u]^\alpha, [v]^\alpha),$$

where h is the Hausdorff metric.

Fact

(1) d_∞ is a metric on $\mathcal{K}_F(X)$.

(2) If X us complete [compact], then $\mathcal{K}_F(X)$ is complete [compact].

(3) $A \in \mathcal{K}(X)$ iff $\chi_A \in \mathcal{K}_F(X)$.

(4) For $A, B \in \mathcal{K}(X)$, $d_\infty(\chi_A, \chi_B) = h(A, B)$.

IFS fuzzyfication

(*) A family $\mathcal{R} = \{\rho_1, ..., \rho_n\}$ of selfmaps of $[0, 1]$ is called an admissible system of grey level maps, if
- ρ_j is nondereasing, usc and $\rho_j(0) = 0$ for every $j = 1, ..., n$;
- $\rho_j(1) = 1$ for some $j = 1, ..., n$.

(*) A fuzzy IFS is a pair $(\mathcal{F}, \mathcal{R})$ that consists of an IFS $\mathcal{F}$ and an admissible system of grey level maps $\mathcal{R}$.

(*) Each fuzzy IFS $(\mathcal{F}, \mathcal{R})$ generates the $\mathcal{F} : \mathcal{K}_F(X) \to \mathcal{K}_F(X)$ defined by

$$\mathcal{F}(u) := \bigcup_{j=1}^{n} \rho_j(f_j(u)) = \max\{\rho_j(f_j(u)) : j = 1, ..., n\}$$

Remark

(1) For every $x \in X$,

$$\mathcal{F}(u)(x) = \max\{\rho_j(u(y)) : j = 1, ..., n,\ y \in f_j^{-1}(x)\}$$

(2) For every $\alpha \in [0, 1]$,

$$[\mathcal{F}(u)]^\alpha = \bigcup_{i=1}^{n} f_i([\rho_i(u)]^\alpha)$$

IFS fuzzyfication

(*) A family $\mathcal{R} = \{\rho_1, ..., \rho_n\}$ of selfmaps of $[0, 1]$ is called an admissible system of grey level maps, if
- ρ_j is nondereasing, usc and $\rho_j(0) = 0$ for every $j = 1, ..., n$;
- $\rho_j(1) = 1$ for some $j = 1, ..., n$.

(*) A fuzzy IFS is a pair $(\mathcal{F}, \mathcal{R})$ that consists of an IFS $\mathcal{F}$ and an admissible system of grey level maps $\mathcal{R}$.

(*) Each fuzzy IFS $(\mathcal{F}, \mathcal{R})$ generates the $\mathcal{F} : \mathcal{K}_F(X) \to \mathcal{K}_F(X)$ defined by

$$\mathcal{F}(u) := \bigcup_{j=1}^{n} \rho_j(f_j(u)) = \max\{\rho_j(f_j(u)) : j = 1, ..., n\}$$

Remark

(1) For every $x \in X$,

$$\mathcal{F}(u)(x) = \max\{\rho_j(u(y)) : j = 1, ..., n, \ y \in f_j^{-1}(x)\}$$

(2) For every $\alpha \in [0, 1]$,

$$[\mathcal{F}(u)]^\alpha = \bigcup_{i=1}^{n} f_i([\rho_i(u)]^\alpha)$$

IFS fuzzyfication

(*) A family $\mathcal{R} = \{\rho_1, ..., \rho_n\}$ of selfmaps of $[0,1]$ is called an admissible system of grey level maps, if
- ρ_j is nondereasing, usc and $\rho_j(0) = 0$ for every $j = 1, ..., n$;
- $\rho_j(1) = 1$ for some $j = 1, ..., n$.

(*) A fuzzy IFS is a pair $(\mathcal{F}, \mathcal{R})$ that consists of an IFS $\mathcal{F}$ and an admissible system of grey level maps $\mathcal{R}$.

(*) Each fuzzy IFS $(\mathcal{F}, \mathcal{R})$ generates the $\mathcal{F} : \mathcal{K}_F(X) \to \mathcal{K}_F(X)$ defined by

$$\mathcal{F}(u) := \bigcup_{j=1}^{n} \rho_j(f_j(u)) = \max\{\rho_j(f_j(u)) : j = 1, ..., n\}$$

Remark

(1) For every $x \in X$,

$$\mathcal{F}(u)(x) = \max\{\rho_j(u(y)) : j = 1, ..., n,\ y \in f_j^{-1}(x)\}$$

(2) For every $\alpha \in [0,1]$,

$$[\mathcal{F}(u)]^\alpha = \bigcup_{i=1}^{n} f_j([\rho_j(u)]^\alpha)$$

IFS fuzzyfication

(*) A family $\mathcal{R} = \{\rho_1, ..., \rho_n\}$ of selfmaps of $[0, 1]$ is called an admissible system of grey level maps, if
- ρ_j is nondereasing, usc and $\rho_j(0) = 0$ for every $j = 1, ..., n$;
- $\rho_j(1) = 1$ for some $j = 1, ..., n$.

(*) A fuzzy IFS is a pair $(\mathcal{F}, \mathcal{R})$ that consists of an IFS $\mathcal{F}$ and an admissible system of grey level maps $\mathcal{R}$.

(*) Each fuzzy IFS $(\mathcal{F}, \mathcal{R})$ generates the $\mathcal{F} : \mathcal{K}_F(X) \to \mathcal{K}_F(X)$ defined by

$$\mathcal{F}(u) := \bigcup_{j=1}^{n} \rho_j(f_j(u)) = \max\{\rho_j(f_j(u)) : j = 1, ..., n\}$$

Remark

(1) For every $x \in X$,

$$\mathcal{F}(u)(x) = \max\{\rho_j(u(y)) : j = 1, ..., n,\ y \in f_j^{-1}(x)\}$$

(2) For every $\alpha \in [0, 1]$,

$$[\mathcal{F}(u)]^\alpha = \bigcup_{i=1}^{n} f_j([\rho_j(u)]^\alpha)$$

IFS fuzzyfication

(*) A family $\mathcal{R} = \{\rho_1, ..., \rho_n\}$ of selfmaps of $[0,1]$ is called an admissible system of grey level maps, if
- ρ_j is nondereasing, usc and $\rho_j(0) = 0$ for every $j = 1, ..., n$;
- $\rho_j(1) = 1$ for some $j = 1, ..., n$.

(*) A fuzzy IFS is a pair $(\mathcal{F}, \mathcal{R})$ that consists of an IFS $\mathcal{F}$ and an admissible system of grey level maps $\mathcal{R}$.

(*) Each fuzzy IFS $(\mathcal{F}, \mathcal{R})$ generates the $\mathcal{F} : \mathcal{K}_F(X) \to \mathcal{K}_F(X)$ defined by

$$\mathcal{F}(u) := \bigcup_{j=1}^{n} \rho_j(f_j(u)) = \max\{\rho_j(f_j(u)) : j = 1, ..., n\}$$

Remark

(1) For every $x \in X$,

$$\mathcal{F}(u)(x) = \max\{\rho_j(u(y)) : j = 1, ..., n,\ y \in f_j^{-1}(x)\}$$

(2) For every $\alpha \in [0,1]$,

$$[\mathcal{F}(u)]^\alpha = \bigcup_{i=1}^{n} f_j([\rho_j(u)]^\alpha)$$

fuzzy version of the Hutchinson–Barnsley theorem

Theorem (Cabrelli et.al 1992, Oliveira and S., 2016)
Assume that X is a complete metric space and $(\mathcal{F},\mathcal{R})$ is a fuzzy IFS consisting of Banach (or weak) contractions. Then there exists the unique $u_\mathcal{F} \in \mathcal{K}_F(X)$ (called the fuzzy attractor) such that $\mathcal{F}(u_\mathcal{F}) = u_\mathcal{F}$.
Moreover, for every $u_0 \in \mathcal{K}_F(X)$, the sequence of iterates $\mathcal{F}^{(k)}(u_0)$ converges to $u_\mathcal{F}$ w.r.t. the metric d_∞.

Remark
If $u_\mathcal{F}$ is a fuzzy attractor, then for every $\alpha \in [0,1]$,

$$[u_\mathcal{F}]^\alpha = \bigcup_{j=1}^{n} f_j([\rho_j(u_\mathcal{F})]^\alpha)$$

Theorem (Oliveira and S., 2016)
In the above frame, set $I = \{i : \rho_j(1) = 1\}$, and let $\mathcal{F}' = \{f_i : i \in I\}$. Then $[u_\mathcal{F}]^0 = A_\mathcal{F}$ and $[u_\mathcal{F}]^1 = A_{\mathcal{F}'}$.

fuzzy version of the Hutchinson–Barnsley theorem

Theorem (Cabrelli et.al 1992, Oliveira and S., 2016)
Assume that X is a complete metric space and $(\mathcal{F}, \mathcal{R})$ is a fuzzy IFS consisting of Banach (or weak) contractions. Then there exists the unique $u_{\mathcal{F}} \in \mathcal{K}_F(X)$ (called the fuzzy attractor) such that $\mathcal{F}(u_{\mathcal{F}}) = u_{\mathcal{F}}$.
Moreover, for every $u_0 \in \mathcal{K}_F(X)$, the sequence of iterates $\mathcal{F}^{(k)}(u_0)$ converges to $u_{\mathcal{F}}$ w.r.t. the metric d_∞.

Remark
If $u_{\mathcal{F}}$ is a fuzzy attractor, then for every $\alpha \in [0,1]$,

$$[u_{\mathcal{F}}]^\alpha = \bigcup_{j=1}^{n} f_j([\rho_j(u_{\mathcal{F}})]^\alpha)$$

Theorem (Oliveira and S., 2016)
In the above frame, set $I = \{i : \rho_j(1) = 1\}$, and let $\mathcal{F}' = \{f_i : i \in I\}$. Then $[u_{\mathcal{F}}]^0 = A_{\mathcal{F}}$ and $[u_{\mathcal{F}}]^1 = A_{\mathcal{F}'}$.

fuzzy version of the Hutchinson–Barnsley theorem

Theorem (Cabrelli et.al 1992, Oliveira and S., 2016)
Assume that X is a complete metric space and $(\mathcal{F}, \mathcal{R})$ is a fuzzy IFS consisting of Banach (or weak) contractions. Then there exists the unique $u_{\mathcal{F}} \in \mathcal{K}_F(X)$ (called the fuzzy attractor) such that $\mathcal{F}(u_{\mathcal{F}}) = u_{\mathcal{F}}$.
Moreover, for every $u_0 \in \mathcal{K}_F(X)$, the sequence of iterates $\mathcal{F}^{(k)}(u_0)$ converges to $u_{\mathcal{F}}$ w.r.t. the metric d_∞.

Remark
If $u_{\mathcal{F}}$ is a fuzzy attractor, then for every $\alpha \in [0, 1]$,

$$[u_{\mathcal{F}}]^\alpha = \bigcup_{j=1}^{n} f_j([\rho_j(u_{\mathcal{F}})]^\alpha)$$

Theorem (Oliveira and S., 2016)
In the above frame, set $I = \{i : \rho_j(1) = 1\}$, and let $\mathcal{F}' = \{f_i : i \in I\}$. Then $[u_{\mathcal{F}}]^0 = A_{\mathcal{F}}$ and $[u_{\mathcal{F}}]^1 = A_{\mathcal{F}'}$.

generalized IFSs

If X is a metric space and $m \in \mathbb{N}$, then we endow the Cartesian product X^m with the maximum metric d_m.

Definiton

(*) A finite family $\mathcal{G}$ of continuous maps from X^m to X will be called a *generalized iterated function system (GIFS) of order m*.

(*) Each GIFS $\mathcal{G} = \{g_1, ..., g_n\}$ generates the map $\mathcal{G} : \mathcal{K}(X)^m \to \mathcal{K}(X)$ by setting

$$\mathcal{G}(K_1, ..., K_m) := \bigcup_{j=1}^{n} g_j(K_1 \times ... \times K_m).$$

(*) A map $g : X^m \to X$ is called *a generalized Banach contraction of order m*, if $\mathrm{Lip}(g) < 1$.

(*) A map $g : X^m \to X$ is called *a generalized weak contraction of order m*, if ... it satisfies weaker contractive condition.

generalized IFSs

If X is a metric space and $m \in \mathbb{N}$, then we endow the Cartesian product X^m with the maximum metric d_m.

Definiton

(*) A finite family $\mathcal{G}$ of continuous maps from X^m to X will be called a *generalized iterated function system (GIFS) of order m*.

(*) Each GIFS $\mathcal{G} = \{g_1, ..., g_n\}$ generates the map $\mathcal{G} : \mathcal{K}(X)^m \to \mathcal{K}(X)$ by setting

$$\mathcal{G}(K_1, ..., K_m) := \bigcup_{j=1}^{n} g_j(K_1 \times ... \times K_m).$$

(*) A map $g : X^m \to X$ is called *a generalized Banach contraction of order m*, if $\text{Lip}(g) < 1$.

(*) A map $g : X^m \to X$ is called *a generalized weak contraction of order m*, if ... it satisfies weaker contractive condition.

generalized IFSs

If X is a metric space and $m \in \mathbb{N}$, then we endow the Cartesian product X^m with the maximum metric d_m.

Definiton

(*) A finite family $\mathcal{G}$ of continuous maps from X^m to X will be called a *generalized iterated function system (GIFS) of order m*.

(*) Each GIFS $\mathcal{G} = \{g_1, ..., g_n\}$ generates the map $\mathcal{G} : \mathcal{K}(X)^m \to \mathcal{K}(X)$ by setting

$$\mathcal{G}(K_1, ..., K_m) := \bigcup_{j=1}^{n} g_j(K_1 \times ... \times K_m).$$

(*) A map $g : X^m \to X$ is called *a generalized Banach contraction of order m*, if $\mathrm{Lip}(g) < 1$.

(*) A map $g : X^m \to X$ is called *a generalized weak contraction of order m*, if ... it satisfies weaker contractive condition.

generalized IFSs

If X is a metric space and $m \in \mathbb{N}$, then we endow the Cartesian product X^m with the maximum metric d_m.

Definiton

(*) A finite family $\mathcal{G}$ of continuous maps from X^m to X will be called a *generalized iterated function system (GIFS) of order m*.

(*) Each GIFS $\mathcal{G} = \{g_1, ..., g_n\}$ generates the map $\mathcal{G} : \mathcal{K}(X)^m \to \mathcal{K}(X)$ by setting

$$\mathcal{G}(K_1, ..., K_m) := \bigcup_{j=1}^{n} g_j(K_1 \times ... \times K_m).$$

(*) A map $g : X^m \to X$ is called *a generalized Banach contraction of order m*, if $\mathrm{Lip}(g) < 1$.

(*) A map $g : X^m \to X$ is called *a generalized weak contraction of order m*, if ... it satisfies weaker contractive condition.

generalized IFSs

If X is a metric space and $m \in \mathbb{N}$, then we endow the Cartesian product X^m with the maximum metric d_m.

Definiton

(*) A finite family $\mathcal{G}$ of continuous maps from X^m to X will be called a *generalized iterated function system (GIFS) of order m*.

(*) Each GIFS $\mathcal{G} = \{g_1, ..., g_n\}$ generates the map $\mathcal{G} : \mathcal{K}(X)^m \to \mathcal{K}(X)$ by setting

$$\mathcal{G}(K_1, ..., K_m) := \bigcup_{j=1}^{n} g_j(K_1 \times ... \times K_m).$$

(*) A map $g : X^m \to X$ is called *a generalized Banach contraction of order m*, if $\mathrm{Lip}(g) < 1$.

(*) A map $g : X^m \to X$ is called *a generalized weak contraction of order m*, if ... it satisfies weaker contractive condition.

generalized IFSs

Theorem (Mihail, Miculescu, S., Swaczyna 2010's)
If X is a complete metric space and $\mathcal{G} = \{g_1, ..., g_n\}$ is a GIFS on X of order m comprising of generalized weak contractions, then there exists the unique $A_\mathcal{G} \in \mathcal{K}(X)$ such that

$$A_\mathcal{G} = \mathcal{G}(A_\mathcal{G}, ..., A_\mathcal{G}) = \bigcup_{j=1}^{n} g_j(A_G \times ... \times A_G)$$

Moreover, for every $K_1, ..., K_m \in \mathcal{K}(X)$, the sequence (K_k) defined by

$$K_{k+m} = \mathcal{G}(K_k, ..., K_{k+m-1}),$$

converges to $A_\mathcal{G}$.

generalized IFSs

Theorem (Mihail, Miculescu, S., Swaczyna 2010's)
If X is a complete metric space and $\mathcal{G} = \{g_1, ..., g_n\}$ is a GIFS on X of order m comprising of generalized weak contractions, then there exists the unique $A_{\mathcal{G}} \in \mathcal{K}(X)$ such that

$$A_{\mathcal{G}} = \mathcal{G}(A_{\mathcal{G}}, ..., A_{\mathcal{G}}) = \bigcup_{j=1}^{n} g_j(A_G \times ... \times A_G)$$

Moreover, for every $K_1, ..., K_m \in \mathcal{K}(X)$, the sequence (K_k) defined by

$$K_{k+m} = \mathcal{G}(K_k, ..., K_{k+m-1}),$$

converges to $A_{\mathcal{G}}$.

GIFS fuzzyfication

(*) For fuzzy set $u_1, ..., u_m : X \to [0,1]$, define $u_1 \times ... \times u_m : X^m \to [0,1]$ by

$$(u_1 \times .... \times u_m)(x_1, ..., x_m) = \min\{u_i(x_i) : i = 1, ..., m\}.$$

(*) A pair $(\mathcal{G}, \mathcal{R})$ will be called a fuzzy GIFS of order m, if $\mathcal{G}$ is a GIFS of order m and $\mathcal{R}$ is an admissible system of grey level maps.

(*) Each fuzzy GIFS $(\mathcal{G}, \mathcal{R})$ generates the map $\mathcal{G} : \mathcal{K}_F(X)^m \to \mathcal{K}_F(X)$ by

$$\mathcal{G}(u_1, ..., u_m) := \bigcup_{j=1}^{n} \rho_j(g_j(u_1 \times ... \times u_m)) = \max\{\rho_j(g_j(u_1 \times ... \times u_m)) : j = 1, ..., n\}$$

Remark

(1) For every $x \in X$,

$$\mathcal{G}(u_1, ..., u_m)(x) = \max\{\rho_j((u_1 \times ... \times u_m)(y)) : j = 1, ..., n,\ y \in f_j^{-1}(x)\})\}$$

(2) For every $\alpha \in [0,1]$,

$$[\mathcal{G}(u_1, ..., u_m)]^\alpha = \bigcup^{n} g_j([\rho_j(u_1 \times ... \times u_m)]^\alpha)$$

GIFS fuzzyfication

(*) For fuzzy set $u_1, ..., u_m : X \to [0,1]$, define $u_1 \times ... \times u_m : X^m \to [0,1]$ by

$$(u_1 \times .... \times u_m)(x_1, ..., x_m) = \min\{u_i(x_i) : i = 1, ..., m\}.$$

(*) A pair $(\mathcal{G}, \mathcal{R})$ will be called a fuzzy GIFS of order m, if $\mathcal{G}$ is a GIFS of order m and $\mathcal{R}$ is an admissible system of grey level maps.

(*) Each fuzzy GIFS $(\mathcal{G}, \mathcal{R})$ generates the map $\mathcal{G} : \mathcal{K}_F(X)^m \to \mathcal{K}_F(X)$ by

$$\mathcal{G}(u_1, ..., u_m) := \bigcup_{j=1}^{n} \rho_j(g_j(u_1 \times ... \times u_m)) = \max\{\rho_j(g_j(u_1 \times ... \times u_m)) : j = 1, ..., n\}$$

Remark

(1) For every $x \in X$,

$$\mathcal{G}(u_1, ..., u_m)(x) = \max\{\rho_j((u_1 \times ... \times u_m)(y)) : j = 1, ..., n, \; y \in f_j^{-1}(x)\})\}$$

(2) For every $\alpha \in [0,1]$,

$$[\mathcal{G}(u_1, ..., u_m)]^\alpha = \bigcup_{j=1}^{n} g_j([\rho_j(u_1 \times ... \times u_m)]^\alpha)$$

GIFS fuzzyfication

(*) For fuzzy set $u_1, ..., u_m : X \to [0,1]$, define $u_1 \times ... \times u_m : X^m \to [0,1]$ by

$$(u_1 \times .... \times u_m)(x_1, ..., x_m) = \min\{u_i(x_i) : i = 1, ..., m\}.$$

(*) A pair $(\mathcal{G}, \mathcal{R})$ will be called a fuzzy GIFS of order m, if $\mathcal{G}$ is a GIFS of order m and $\mathcal{R}$ is an admissible system of grey level maps.

(*) Each fuzzy GIFS $(\mathcal{G}, \mathcal{R})$ generates the map $\mathcal{G} : \mathcal{K}_F(X)^m \to \mathcal{K}_F(X)$ by

$$\mathcal{G}(u_1, ..., u_m) := \bigcup_{j=1}^{n} \rho_j(g_j(u_1 \times ... \times u_m)) = \max\{\rho_j(g_j(u_1 \times ... \times u_m)) : j = 1, ..., n\}$$

Remark

(1) For every $x \in X$,

$$\mathcal{G}(u_1, ..., u_m)(x) = \max\{\rho_j((u_1 \times ... \times u_m)(y)) : j = 1, ..., n,\ y \in f_j^{-1}(x)\})\}$$

(2) For every $\alpha \in [0,1]$,

$$[\mathcal{G}(u_1, ..., u_m)]^\alpha = \bigcup^{n} g_j([\rho_j(u_1 \times ... \times u_m)]^\alpha)$$

GIFS fuzzyfication

(*) For fuzzy set $u_1, ..., u_m : X \to [0,1]$, define $u_1 \times ... \times u_m : X^m \to [0,1]$ by

$$(u_1 \times .... \times u_m)(x_1, ..., x_m) = \min\{u_i(x_i) : i = 1, ..., m\}.$$

(*) A pair $(\mathcal{G}, \mathcal{R})$ will be called a fuzzy GIFS of order m, if $\mathcal{G}$ is a GIFS of order m and $\mathcal{R}$ is an admissible system of grey level maps.

(*) Each fuzzy GIFS $(\mathcal{G}, \mathcal{R})$ generates the map $\mathcal{G} : \mathcal{K}_F(X)^m \to \mathcal{K}_F(X)$ by

$$\mathcal{G}(u_1, ..., u_m) := \bigcup_{j=1}^{n} \rho_j(g_j(u_1 \times ... \times u_m)) = \max\{\rho_j(g_j(u_1 \times ... \times u_m)) : j = 1, ..., n\}$$

Remark

(1) For every $x \in X$,

$$\mathcal{G}(u_1, ..., u_m)(x) = \max\{\rho_j((u_1 \times ... \times u_m)(y)) : j = 1, ..., n, \ y \in f_j^{-1}(x)\})\}$$

(2) For every $\alpha \in [0,1]$,

$$[\mathcal{G}(u_1, ..., u_m)]^\alpha = \bigcup_{j=1}^{n} g_j([\rho_j(u_1 \times ... \times u_m)]^\alpha)$$

GIFS fuzzyfication

(*) For fuzzy set $u_1, ..., u_m : X \to [0,1]$, define $u_1 \times ... \times u_m : X^m \to [0,1]$ by

$$(u_1 \times .... \times u_m)(x_1, ..., x_m) = \min\{u_i(x_i) : i = 1, ..., m\}.$$

(*) A pair $(\mathcal{G}, \mathcal{R})$ will be called a fuzzy GIFS of order m, if $\mathcal{G}$ is a GIFS of order m and $\mathcal{R}$ is an admissible system of grey level maps.

(*) Each fuzzy GIFS $(\mathcal{G}, \mathcal{R})$ generates the map $\mathcal{G} : \mathcal{K}_F(X)^m \to \mathcal{K}_F(X)$ by

$$\mathcal{G}(u_1, ..., u_m) := \bigcup_{j=1}^{n} \rho_j(g_j(u_1 \times ... \times u_m)) = \max\{\rho_j(g_j(u_1 \times ... \times u_m)) : j = 1, ..., n\}$$

Remark

(1) For every $x \in X$,

$$\mathcal{G}(u_1, ..., u_m)(x) = \max\{\rho_j((u_1 \times ... \times u_m)(y)) : j = 1, ..., n,\ y \in f_j^{-1}(x)\})\}$$

(2) For every $\alpha \in [0,1]$,

$$[\mathcal{G}(u_1, ..., u_m)]^\alpha = \bigcup_{j=1}^{n} g_j([\rho_j(u_1 \times ... \times u_m)]^\alpha)$$

fuzzy version of the Hutchinson–Barnsley theorem for GIFSs

Theorem (Oliveira and S., 2016)
Assume that X is a complete metric space and $(\mathcal{G}, \mathcal{R})$ is a fuzzy GIFS consisting of generalized weak contractions. Then there exists the unique $u_{\mathcal{G}} \in \mathcal{K}_F(X)$ (called the fuzzy attractor) such that $\mathcal{G}(u_{\mathcal{G}}, ..., u_{\mathcal{G}}) = u_{\mathcal{G}}$. Moreover, for every $u_1, ..., u_m \in \mathcal{K}_F(X)$, the sequence of iterates (u_k) defined by

$$u_{m+k} = \mathcal{G}(u_k, ..., u_{k+m-1}),$$

converges to $u_{\mathcal{G}}$ w.r.t. the metric d_∞.

Remark
If $u_{\mathcal{G}}$ is a fuzzy attractor, then for every $\alpha \in [0,1]$,

$$[u_{\mathcal{G}}]^\alpha = \bigcup_{j=1}^{n} g_j([\rho_j(u_{\mathcal{G}} \times ... \times u_{\mathcal{G}})]^\alpha)$$

Theorem (Oliveira and S., 2016)
In the above frame, set $I = \{i : \rho_j(1) = 1\}$, and let $\mathcal{G}' = \{g_i : i \in I\}$. Then $[u_{\mathcal{G}}]^0 = A_{\mathcal{G}}$ and $[u_{\mathcal{G}}]^1 = A_{\mathcal{G}'}$.

fuzzy version of the Hutchinson–Barnsley theorem for GIFSs

Theorem (Oliveira and S., 2016)
Assume that X is a complete metric space and $(\mathcal{G},\mathcal{R})$ is a fuzzy GIFS consisting of generalized weak contractions. Then there exists the unique $u_{\mathcal{G}} \in \mathcal{K}_F(X)$ (called the fuzzy attractor) such that $\mathcal{G}(u_{\mathcal{G}},...,u_{\mathcal{G}}) = u_{\mathcal{G}}$. Moreover, for every $u_1,...,u_m \in \mathcal{K}_F(X)$, the sequence of iterates (u_k) defined by

$$u_{m+k} = \mathcal{G}(u_k,...,u_{k+m-1}),$$

converges to $u_{\mathcal{G}}$ w.r.t. the metric d_∞.

Remark
If $u_{\mathcal{G}}$ is a fuzzy attractor, then for every $\alpha \in [0,1]$,

$$[u_{\mathcal{G}}]^\alpha = \bigcup_{j=1}^{n} g_j([\rho_j(u_{\mathcal{G}} \times ... \times u_{\mathcal{G}})]^\alpha)$$

Theorem (Oliveira and S., 2016)
In the above frame, set $I = \{i : \rho_j(1) = 1\}$, and let $\mathcal{G}' = \{g_i : i \in I\}$. Then $[u_{\mathcal{G}}]^0 = A_{\mathcal{G}}$ and $[u_{\mathcal{G}}]^1 = A_{\mathcal{G}'}$.

fuzzy version of the Hutchinson–Barnsley theorem for GIFSs

Theorem (Oliveira and S., 2016)
Assume that X is a complete metric space and $(\mathcal{G}, \mathcal{R})$ is a fuzzy GIFS consisting of generalized weak contractions. Then there exists the unique $u_{\mathcal{G}} \in \mathcal{K}_F(X)$ (called the fuzzy attractor) such that $\mathcal{G}(u_{\mathcal{G}}, ..., u_{\mathcal{G}}) = u_{\mathcal{G}}$. Moreover, for every $u_1, ..., u_m \in \mathcal{K}_F(X)$, the sequence of iterates (u_k) defined by

$$u_{m+k} = \mathcal{G}(u_k, ..., u_{k+m-1}),$$

converges to $u_{\mathcal{G}}$ w.r.t. the metric d_∞.

Remark
If $u_{\mathcal{G}}$ is a fuzzy attractor, then for every $\alpha \in [0, 1]$,

$$[u_{\mathcal{G}}]^\alpha = \bigcup_{j=1}^{n} g_j([\rho_j(u_{\mathcal{G}} \times ... \times u_{\mathcal{G}})]^\alpha)$$

Theorem (Oliveira and S., 2016)
In the above frame, set $I = \{i : \rho_j(1) = 1\}$, and let $\mathcal{G}' = \{g_i : i \in I\}$. Then $[u_{\mathcal{G}}]^0 = A_{\mathcal{G}}$ and $[u_{\mathcal{G}}]^1 = A_{\mathcal{G}'}$.

acknowledgments

Thank You For Your Attention!

References

[1] Carlos A. Cabrelli, Bruno Forte, Ursula M. Molter, and Edward R. Vrscay. Iterated fuzzy set systems: a new approach to the inverse problem for fractals and other sets. J. Math. Anal. Appl., 171 (1992), no. 1, 79–100.

[2] Oliveira, Elismar R.; Strobin, Filip; Fuzzy attractors appearing from GIFZS. Fuzzy Sets Syst. 331 (2018), 131–156.