Differentiability and Haar-smallness

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Haar-small sets

X – abelian Polish group.
$\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ – a semi-ideal ($A \in \mathcal{I} \ \wedge \ B \subseteq A \implies B \in \mathcal{I}$).

Definition (Banakh, Głąb, Jabłońska, Swaczyna)

$A \subseteq X$ is Haar-$\mathcal{I}$ ($A \in \mathcal{HI}$) if there are a Borel hull $B \supseteq A$ and a continuous map $f : 2^\omega \to X$ such that $f^{-1}[B + x] \in \mathcal{I}$ for all $x \in X$.

$\mathcal{I}$	$\mathcal{HI}$
$\mathcal{N}$	Haar-null
$\mathcal{M}$	Haar-meager
$[2^\omega]^{\leq\omega}$	Haar-countable
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- $\mathcal{HI}$ is a translation-invariant semi-ideal (for each $\mathcal{I}$).
- Haar-n $\subseteq$ Haar-$(n+1)$ $\subseteq$ Haar-finite $\subseteq$ Haar-countable $\subseteq$ $\mathcal{HN} \cap \mathcal{HM}$

Theorem (K.; Banakh, Głąb, Jabłońska, Swaczyna)

- *All countable sets are Haar-1.*
- *There is an uncountable Haar-1 set.*
- *Cantor set is Haar-2, but not Haar-1.*

Theorem (K.)

Let $X = \mathbb{R} \times Y$ or $C[0,1]$.

- *None of the above inclusions can be reversed.*
- *Haar-finite sets do not form an ideal.*

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For $f \in C[0,1]$ by $D(f)$ we denote the set of all points $x \in [0,1]$ at which f is differentiable.

f is somewhere differentiable ($f \in \mathcal{SD}[0,1]$) if $D(f) \neq \emptyset$.

Theorem (Banach)

$\mathcal{SD}[0,1]$ is meager in $C[0,1]$.

Theorem (Hunt)

$\mathcal{SD}[0,1]$ is Haar-null in $C[0,1]$.

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Multidimensional case

$f \in C[0,1]^k$ is somewhere differentiable ($f \in \mathcal{SD}[0,1]^k$) if it is differentiable at some point along some vector.

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Definition

$A \subseteq X$ is thick if for any compact set $K \subseteq X$ there is $x \in X$ with $K + x \subseteq A$

A is thick $\iff \forall_{\mathcal{I} \neq \mathcal{P}(2^\omega)}$ A is not Haar-$\mathcal{I}$.

Proposition (K.-Wołoszyn)

If $k \geq 2$ then $\mathcal{SD}[0,1]^k$ is thick in $C[0,1]^k$.

Actually, for each vector $v \in \mathbb{R}^k$ the set of functions differentiable along v at $\mathfrak{c}$ many points is thick.

Problem

Is the set $C[0,1]^2 \setminus \mathcal{SD}[0,1]^2$ (of nowhere differentiable functions on $[0,1]^2$) thick in $C[0,1]^2$?

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Differentiability

$\mathcal{I}$ – "nice" σ-ideal $\quad \mathcal{D}(\mathcal{A}) = \{f \in C[0,1] : D(f) \in \mathcal{A}\}$

$\mathcal{A} =$	$\mathcal{D}(\mathcal{A}) \in$	$\mathcal{D}(\mathcal{A}) \notin$
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Theorem (Hunt)

The set $\mathcal{SD}[0,1]$ of somewhere differentiable functions is Haar-null.

Actually, Hunt showed that it is Haar-$\mathcal{E}$, where $\mathcal{E} \subsetneq \mathcal{N} \cap \mathcal{M}$ is the σ-ideal generated by closed null sets.

Note that $\mathcal{D}(\mathcal{I} \setminus \{\emptyset\}) \subseteq \mathcal{SD}[0,1]$.

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A σ-ideal $\mathcal{I}$ is ccc, if every family of pairwise disjoint Borel sets not belonging to $\mathcal{I}$ is countable.

Theorem (K., Wołoszyn)

If $\mathcal{I}$ is ccc, then the set $\mathcal{D}(\mathcal{I}^c)$ of functions differentiable on an $\mathcal{I}$-positive set is Haar-countable.

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Theorem (Banakh, Głąb, Jabłońska, Swaczyna)

If $A - A$ is meager, then A is Haar-1.

$$D(f), D(g) \in \mathcal{I}^* \implies f - g \in \mathcal{SD}[0,1].$$

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Theorem (Banakh, Głąb, Jabłońska, Swaczyna)

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