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Uogólnienie kraty sytuacji elementarnych Wolniewicza.
Podejście topologiczne

4 Warsztaty z Analizy Rzeczywistej
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1. What is topological ontology?

We usually describe ontology in accordance with Greek philosophy (Aristotle) as a branch of philosophy concerned with an object as object or being as being. It means that we try to define what is or what exists, i.e. being and some fundamental properties of being. Philosophers, ontologists pointed to different collections of categories constituting the fundamental characteristics of being. For example, in Aristotle's papers, we find a system of categories in which primary substance is a distinguished category and distinct from nine others: qualities, quantities, relations, place, time, actions and affections. Having these nine categories we can predicate: Socrates (primary substance) is righteous (quality), gives a lecture for his students (relation), on Athenian agora (place), at noon (time) etc.

We can also propose, for example, Husserl's categories concerning his *formal ontology* i.e. ontology of an object as object (object in general). Husserl suggested the following categories (concepts or terms): object, state of affairs, property, relation, number, unit, plurality etc. Apart from them, Husserl introduced the so called *material ontology* with three general regions, such as: nature (including physical objects and events but also the world of what is alive), culture (artefacts, social entities and values), and consciousness (cf. Husserl, 1913).

In my monograph (Kaczmarek, 2008) I proposed an array of terms and notions that are important in the ontological investigations. I pointed out the following levels and the relevant terms:

a) the level of individuals – individuals, properties, essential and attributive property, positive and negative property, complete object, extension of idea, but also state of affairs, facts, relations
b) the level of ideas – general object, species, genera, hierarchy of general objects, species difference, property of idea and property given in a content of idea,
c) the level of concepts – concept, the structure of concepts, the content of a concept, positive and negative content, extension of a concept.

I present the definitions of terms and notions in question and some theorems in set-theoretical language (cf. Kaczmarek, 2008, 2008b).

But now the problem is: what is topological ontology? Here we are interested not in formal ontology in general, but in such a kind of ontology which tries to use topological concepts, topological theorems and structures, and – perhaps – also a topological point of view[1]. In a private conversation, my younger colleague, Bartłomiej Skowron, suggested to me to speak about modelling of ontological concepts, objects, and theorems with topological ones as far as topological ontology is concerned. We can find this perspective in Salamucha's papers. This Polish analytic philosopher pointed out the so called "geometric point of view" because, in his opinion, after the discovery of Euclidean geometry this point of view would be dominating in science and philosophy (cf. Salamucha, 1946). Let us emphasize that this approach style of work was already present in Aristotle's metaphysics.

[1]The term "topological ontology" and "topological philosophy" (in Polish: "ontologia topologiczna" and "topologiczna filozofia") was proposed by pr. Mirosław Szatkowski and Bartłomiej Skowron.

2. Wolniewicz’ Lattices of (Elementary) Situations

Co to są sytuacje?

1) program Russell’a i Wittgensteina: struktura świata jest izomorficzna ze strukturą języka logiki (pierwszego rzędu)

2) przykłady sytuacji:

a) Warszawa leży nad Wisłą

b) W Konopnicy odbywają się teraz 4 warsztaty z analizy rzeczywistej

c) Jan kocha Annę, Anna kocha Pawła, Paweł kocha Marię a Maria kocha Jana

Axioms.

(Ax 1) ES = PES $\cup$ {0, λ}.

(Ax 2) Let $\leq$ be a partially order relation on ES, where 0 is the smallest situation and λ the greater one. Then:

for any $x \in$ ES: $0 \leq x \leq \lambda$.

(Ax 3) For any set A of ES there exists s such that – m :

s = sup$A$,

(Wolniewicz introduces here two operations: $\cup$ – supremum of s and s' and $\cap$ – infimum of s and s').

(Ax 4) For any $x, y, z \in$ ES: $x \leq y \leq z \Rightarrow \exists_{y' \in SE} (x = y \cap y'$ oraz $z = y \cup y')$. (complementary)

(Ax 5) For any $x, y, z \in$ ES:

(a) $(x \cup y \neq s_1 \wedge x \cup z \neq s_1) \Rightarrow (x \cup y) \cap (x \cup z) \leq x \cup (y \cap z)$,

(b) $y \cup z \neq s_1 \Rightarrow x \cap (y \cup z) \leq (x \cap y) \cup (x \cap z)$. (distributivity)

(Ax 6) Let ES' = ES – $\{\lambda\}$. Then: $\forall_{s \in SE'} \exists_{w \in \mathrm{Max}(ES')}\ s \leq w)$.

(Ax 7) (Separability) For any $x, y \in$ ES: $x \neq y \Rightarrow \exists_{w \in SP} ((x \leq w \wedge \sim y \leq w) \vee (\sim x \leq w \wedge y \leq w))$.

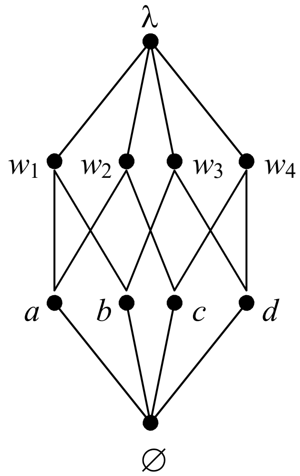
(Ax 8) (Atomicity) Any lattice of elementary situations has non-empty set of atoms i. e.: $\forall_{x \in \mathrm{E}S}$ $\exists_{A \subset SA}$ $(x = \sup A))$.

(Ax 9) For any $x, y \in ES$: $x \cup y = \lambda \Rightarrow \exists_{a, a' \in SA} (a \leq x \ \wedge \ a' \leq y \ \wedge \ a \cup a' = \lambda)$.

(Ax 10) $\forall_{x, y, z \in SA} ((x \cup z = \lambda \wedge y \cup z = \lambda) \Rightarrow (x = y \vee x \cup y = \lambda)))$.

(Ax 11) The number of dimensions is finite.

Let us present some example of Wolniewicz' lattice. We can conceive of a given lattice of elementary situations by the following picture:



Let us explain some matters. The set $\{a, b, c, d\}$ is a set of atoms (simple state of affairs; simple and compound states are called situations). We can interpret them as “it’s cold”, “it’s wet”, “it’s dry” and “it’s warm”, respectively. $\{a, d\}$ and $\{b, c\}$ are two logical dimensions of temperature and moisture. Wolniewicz assumes that the number of dimensions is finite but the numbers of atoms in a given dimension D is arbitrary (finite or infinite). Every lattice of elementary situations SE comprises at least two improper ones: the impossible situation λ, and the empty one $\varnothing$ ($\varnothing \neq \lambda$). Wolniewicz assumes also: the set $SE' = SE - \{\lambda\}$is a set of possible elementary situations, and $SE'' = SE - \{\varnothing, \lambda\}$ is that of the contingent ones. Next, the set $\{w_1, w_2, w_3, w_4\}$ is a set of possible worlds (a logical space). Of course, for any situation x: $\varnothing \leq x \leq \lambda$.

3. Topology

Let us bring back some basic notions of general topology.

DEFINITION 1. Let X be a set (not necessarily nonempty) and T_X a family of subsets of X. A pair (X,T_X) is a topology or a topological space on X, if the following conditions are fulfilled:

a) $\varnothing \in T_X$ and $X \in T_X$,
b) A union of sets from T_X is a set of T_X,
c) A finite intersection of sets from T_X is a set of T_X.

Examples of topologies.

$\tau 1$. If $X = \varnothing$, then $(\varnothing, \{\varnothing\})$ is topological space.

$\tau 2$. If $X = \{1; 2\}$, then $(X, \{\varnothing, \{1\}, X\})$ is a topological space. It is known as the *Sierpiński's space*.

τ3. If $X = R$, R is the set of real numbers, and any set of T_R is a union of sets in form $(r_1; r_2)$, for $r_i \in R$, then (R, T_R) is topological space called *natural topology on R*.

τ4. If $X = R$ and $\varnothing \neq A \subset X$, then $(X, \{\varnothing, A, X - A, X\})$ is topological space.

τ5. For any set X the *discrete topology* on X is the topology T_d such that $T_d = \{U: U \subseteq X\}$, so the collection of open sets of T_d equals the power set of X, i.e. $T_d = P(X)$. Next, the *indiscrete topology* (or *trivial topology*) on X is the topology $T_{triv} = \{\varnothing, X\}$.

DEFINITION 2. Any element of T_X we call an open set. If A is open set, then X – A is called a closed set of the topological space T_X. Of course X and $\varnothing$, are open and closed in each topological space at the same time.

DEFINITION 3. A collection $\mathcal{B}$ of open sets of a topological space (X,T_X) is called a basis if each open set in X can be represented as a union of elements of $\mathcal{B}$.

DEFINITION 4. Let (X,T_X) be a topological space. A collection S of open sets is called a subbasis if each open set in a basis of (X,T_X) can be represented as a union of finite intersections of elements of S.

Remark 1 (comp. Kulpa [2013], p. 49). For any family $T'_X \subseteq 2^X$, exists the least topology T_X which have the following properties: 1) $S \subseteq T_X$, 2) for any $T'_X \subseteq 2^X$, if $S \subseteq T'_X$, then $T_X \subseteq T'_X$.

Proof. Really, the least topology T_X which includes the family S is the topology defined in 3 stages:

a) subbasis: $S \cup \{\emptyset, X\}$,

b) basis: $\mathcal{B} = \{A_1 \cap ... \cap A_n: A_i \in S \cup \{\emptyset, X\}$, for $i = 1, ..., n, n \in \omega\}$,

c) topology: $T_X = \{\bigcup W : W \subseteq \mathcal{B}\}$.

The remark given above is important for understanding ideas and Wittgenstein topology that will be introduced below.

Examples: subbasis and basis

$\tau 6$. A family $\mathcal{B} = \{(r_1; r_2), \text{ for } r_1, r_2 \in Q\} \cup \{\varnothing\}$ is one of the basis of natural topology.
$\tau 7$. A family $S = \{(-\infty; r_2), \text{ for } r_2 \in Q\} \cup \{(r_1; \infty), \text{ for } r_1 \in Q\}$ is one of the subbasis of the natural Euclidean topology.
$\tau 8$. Let us consider the subbasis $S = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5, 6\}\} \cup \{\varnothing\}$. Then a family:

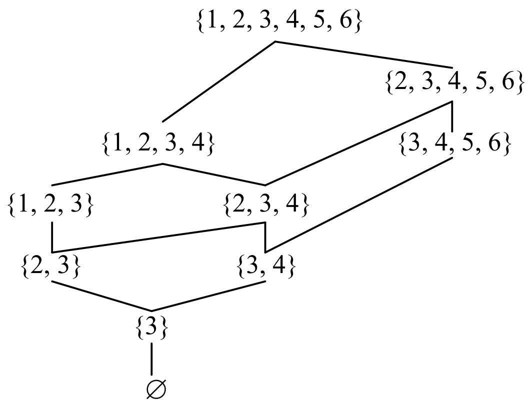
$$\mathcal{B} = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5, 6\}, \{2, 3\}, \{3\}, \{3, 4\}\} \cup \{\varnothing\}$$

is the basis and

$$T_{\{1, 2, 3, 4, 5, 6\}} = \{\varnothing, \{1, 2, 3, 4, 5, 6\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5, 6\}, \{2, 3\}, \{3\}, \{3, 4\}, \{1, 2, 3, 4\}, \{2, 3, 4, 5, 6\}\}$$

is the topology generated by the given subbasis and basis.

Some connection between topologies and lattices we will use later, so let us remark that the topology $T_{\{1, 2, 3, 4, 5, 6\}}$ can be depicted as the lattice structure:



4. Discrete topology of situations

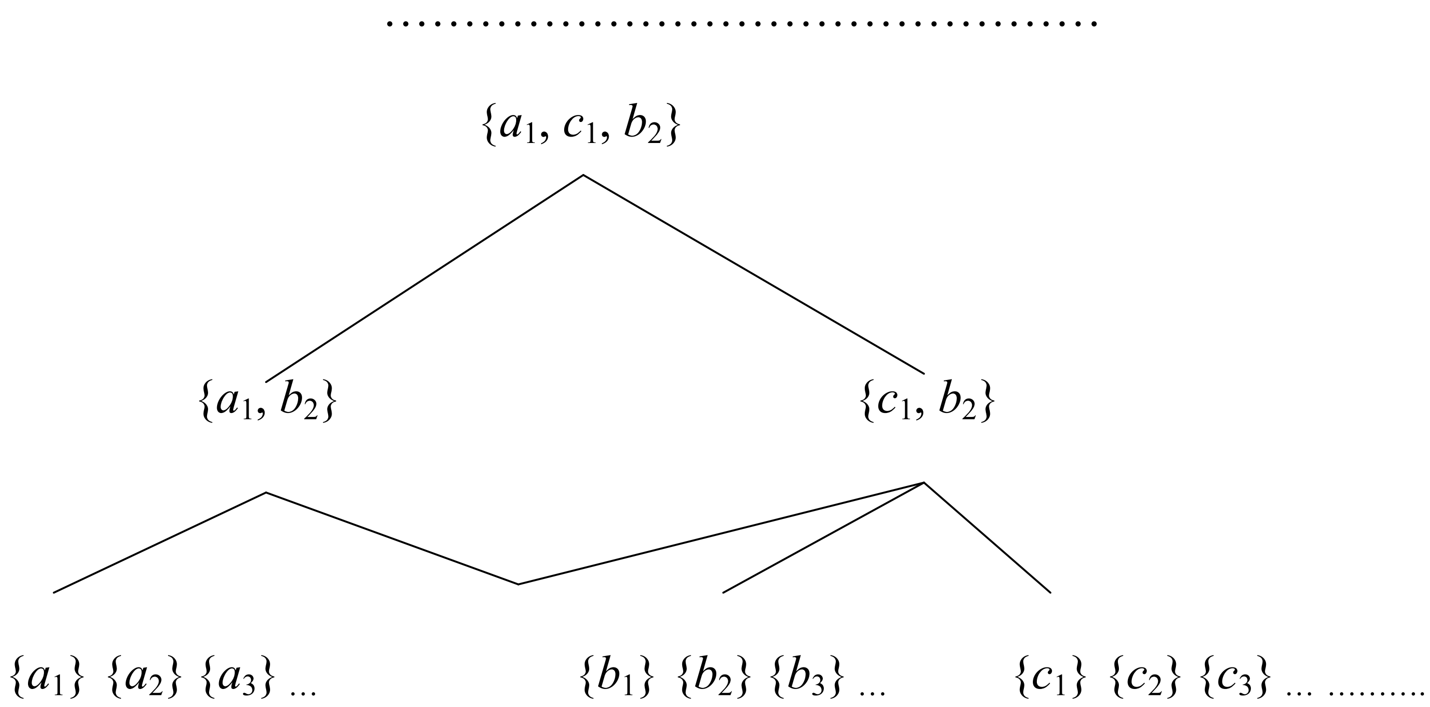
Let us define lattice of situations from the bottom up.

Let $A = \{A_1, A_2, A_3, \ldots..\}$ be any countable family of nonempty sets and for any A_i, A_j, $A_i \cap A_j = \varnothing$ (now we reject Wolniewicz's axiom that the number of dimensions is finite). Each A_i is understood as a set of atomic situations. A_i is of course a dimension of a lattice and a set of incompatible atoms (why? because for any $\{a\}$ and $\{b\}$ from A_i $\{a\} \cup \{b\} = \lambda$; comp. Point 5, below). Now let us take into account a function c: $\mathcal{N} \rightarrow \bigcup_{k=1}^{\infty} A_k$, such that $c(k) \in A_k$. Then we fix, for the given c:

$$\mathcal{B}_c = \{\{ c(k)\}: k \in \aleph\} \cup \{\varnothing\}.$$

FACT 1. A pair (X, T_X), where $X = \bigcup_{k=1}^{\infty}\{ c(k)\}$ and for any $A \in T_X$, $A = \bigcup B$ for $B \in \mathcal{B}_c$, is a topological space. A space fixed by the fact I propose to call Wittgenstein's topology.

The Wittgenstein's topology may be visualised as:



So, we can obtain:

FACT 2. A union of all Wittgenstein's topologies is a lattice (with $\varnothing$ as a empty situation and $\bigcup(A \cup B \cup C \ldots.)$ as λ; let us call it *LWT*).

DEFINITION 5. Elements from A, B, C, ... are called atoms of the lattice. Elements of the form $w = \{c(k)\}: k \in \aleph\}$, for any c, are possible worlds.

FACT 3. All axioms of Wolniewicz's lattice are fulfilled in *LWT* (except the condition that the number of dimensions is finite).

FACT. **4**. In *LWT* there are exist many topologies that are homeomorphic with the given Wittgenstein's topology.

5. Dependence and independence of state of affairs. Compatible and Incompatible ones.

Let us remark that atomic elementary situations belonging to different logical dimensions are independent of each other in a Wittgensteinian sense. I propose to define:

DEFINITION 6. Let ES be a non-empty set of elementary situations from LWT. ES is independent iff $\sup ES \neq \lambda$ and for any A, $B \in ES$: $A = B$ or $A \cap B = \varnothing$.

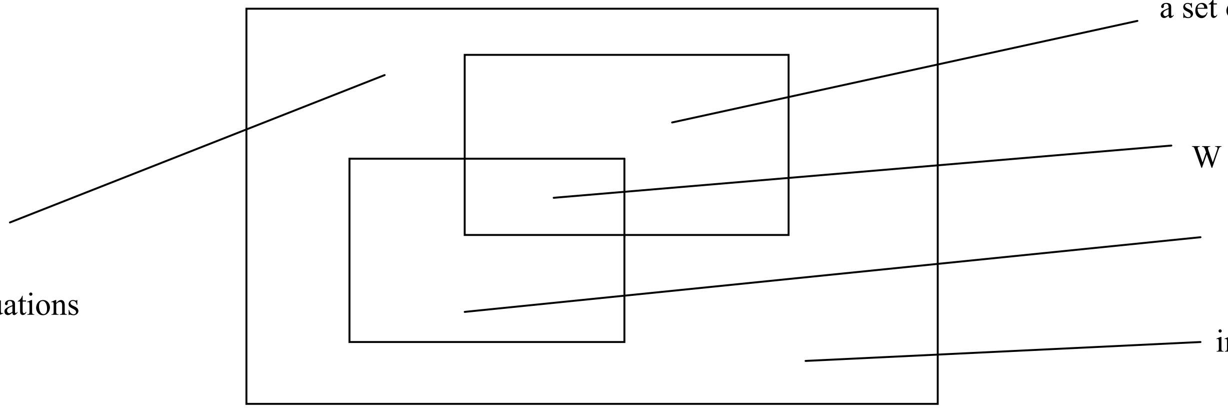
FACT 5. Any two situations A, B are independent iff $A \cup B \neq \lambda$ and $A \cap B = \varnothing$.

DEFINITION 7. Any two contingent situations A and B are called incompatible ones iff $\mathrm{A} \cup \mathrm{B} = \lambda$.

FACT 6. Let IN be a set of independent situations and COM a set of compatible ones. Then:
(i) $IN \cap COM \neq \varnothing$,
(ii) $IN - COM \neq \varnothing$,
(iii) $COM - IN \neq \varnothing$.

Picture

A set of (elementary) situations:



Situations

a set of W-independent situations, for example:

$\varnothing$ and $\{a, b\}$

W – independent and compatible: $\{a, b\}$ and $\{c, d\}$

compatible and not independent situations: $\{a\}$ and $\{a, b\}$

incompatible and not independent: $\{a_1\}$ i $\{a_2\}$ – (objects from one D)

6. Non-atomistic topology

When Wittgenstein was asked about an example of simple state of affairs he answered supposedly: I don't know. So, it is possible that he thought also about non-atomistic collection of situations (states of affairs). Let us look on this kind of collection.

W know that any subbase $\{\varnothing, X, ...\}$ generates base and the smallest topological space. So, let us use that fact.

DEFINITION 8. Let (X, T_X) be any topological space. For any $A \subset X$ consider subspaces on A such that for any $A_i, A_j, \subset X$, for $i \neq j$, $A_i \not\subset A_j$ and $A_j \not\subset A_i$.

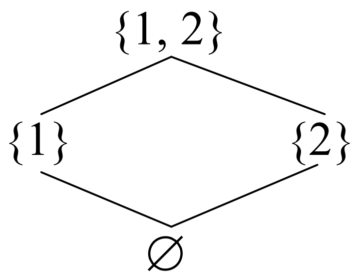
Remark. We are considering topological subspaces to generate the so-called possible worlds.

FACT 7. A union $\bigcup_i A_i$ of topological spaces with $\lambda = X$ as unit is a lattice (we call it non-atomic lattice L).

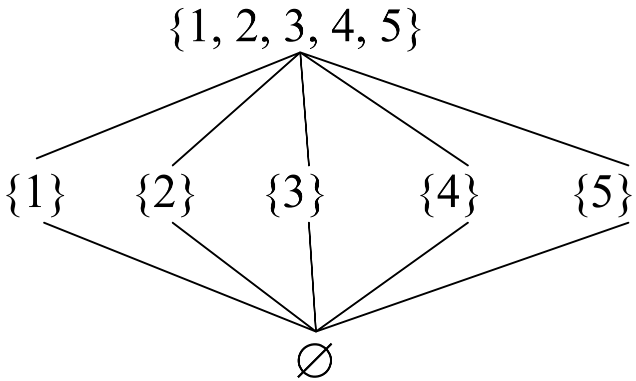
REMARK. Non-atomicity means: for any $A \neq \varnothing$, a set $\{C: \varnothing < C < A\}$ is non-empty.

EXAMPLES

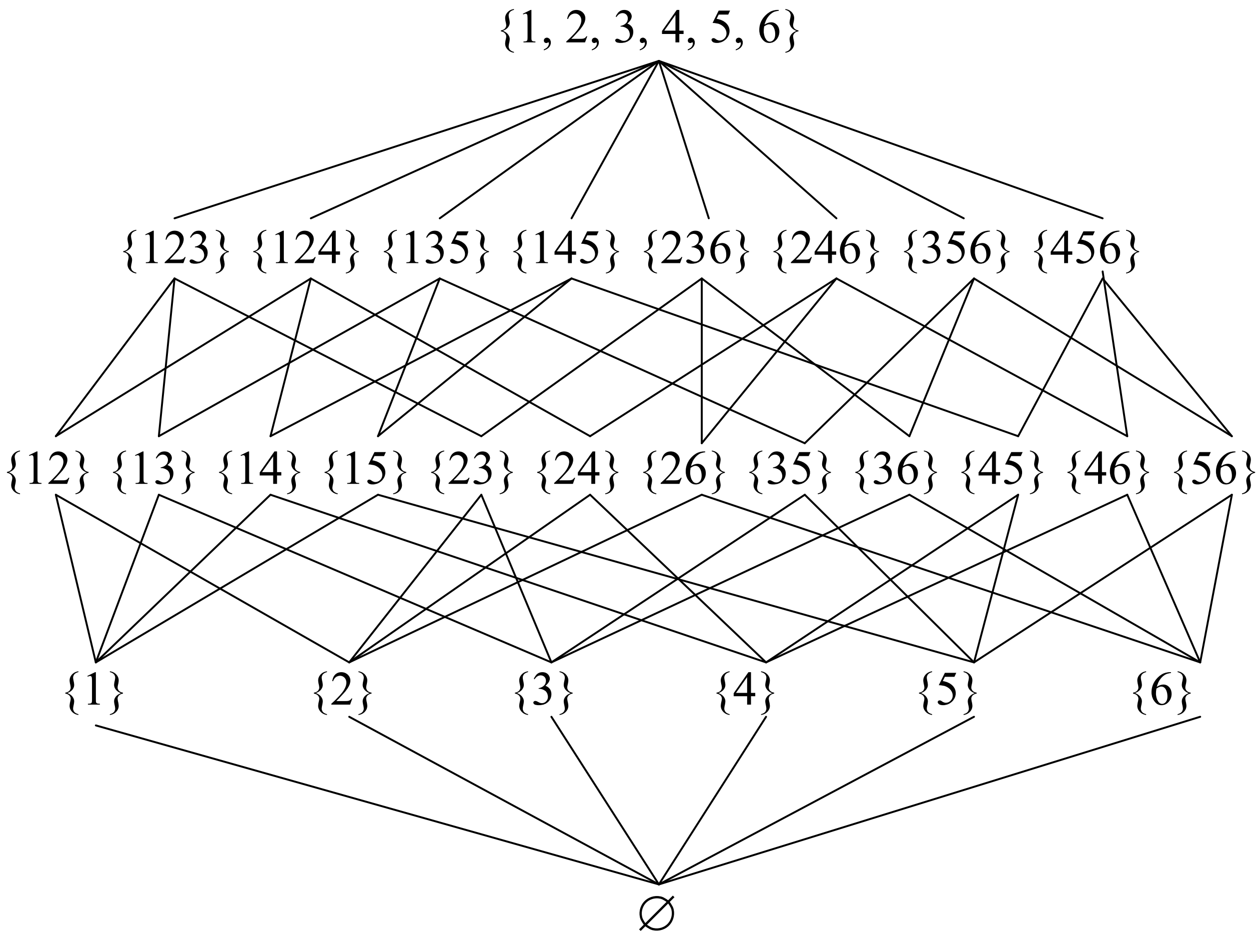
1) Sierpiński's topology; considering topology $(\{1, 2\}, \{\varnothing, \{1\}, \{1,2\})$ and two subspaces on $\{1\}$ and $\{2\}$ we obtain the lattice:



2) let take into account the topological space $(\{1, 2, 3, 4, 5\}, \{\varnothing, \{1\}, \{1, 2, 3, 4, 5\}\})$. Then considering subspaces on $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$ we obtain:



3)



It is a lattice with signature (2, 2, 2); $D_1 = \{1, 6\}$, $D_2 = \{2, 5\}$, $D_3 = \{3,4\}$

4) Let take into account $X = (0; 1)$ and any $r \in (0; 1)$. Now, consider topological subspaces on $X - \{r\}$ having all sets A such that $A = (X - \{r\}) \cap U$, where U is an open set of natural topological space on $(0; 1)$. Let us fix $\lambda = (0; 1)$ and the topology on $X - \{r\}$ as T_r.

FACT 8. A union $\bigcup_r T_r$ of topological spaces with $\lambda = (0; 1)$ as unit and $r \in (0; 1)$ is a lattice.

It is evident that for any r a set $(0; 1) - \{r\}$ plays a role of possible world but in the lattice we have no atoms. Remark also that a set $\{a\} \subset (0; 1)$ is not an atom, because for any topological space T_r: $\{a\} \notin T_r$.

Proof. If $A = B$, the proof is evident. Let $A \neq B$. Then 1) $A \cup B = (0; 1)$ or 2) $A \cup B \subset (0; 1) = \lambda$. So, in the case of 1) supremum of A and B is λ and infimum is an open set from T_r. In the case 2) there exists r such that $r \notin A \cup B$ and then $A, B \in T_r$. Hence, $A \cup B \in T_r$ and $A \cap B \in T_r$. ■

REMARKS.

1. The lattice L given above is (semi)atomistic, i. e.: for any $A \in L$ there exist a family B of elements such that: $A = \bigcup_s B_s$, for $B_S \in B$.
2. Let $\bigcup_i A_i$ with $\lambda = X$ be a non-atomic lattice L. For any $a \in A_i$, $\{a\} \notin L$. We can represent $\{a\}$ as a compliment of an open set U to A_i (A_i is a domain of topological subspace on A_i).
3. Let $\bigcup_i A_i$ with $\lambda = X$ be a non-atomic lattice L. For any A, B $\in L$, if A $\neq$ B then A and B are separated.
4. The concepts of compatible situations and W-independence ones are introduced in L in the same way.

Example. In $\bigcup_r T_r$ 1) two situations A and B are incompatible iff $A \cup B = (0; 1)$,
2) two situations A and B are W-independent iff $A \cap B = \varnothing$.