Haar-open sets: the right way of generalizing the Steinhaus Sum Theorem to non-locally compact groups?

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Lviv & Kielce

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The Steinhaus sum and difference theorems

Theorem (Steinhaus, 1920)

For any measurable subsets $A, B \subset \mathbb{R}$ of positive Lebesgue measure

(+) *the sum-set $A + B := \{a + b : a \in A,\ b \in B\}$ has non-empty interior in $\mathbb{R}$ and*

(−) *the difference set $A - A := \{a - b : a, b \in A\}$ is a neighborhood of zero in $\mathbb{R}$.*

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Theorem (Weil, 1965)

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What about non-locally compact groups?

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Haar-null sets

Definition (Christensen, 1972)

A Borel subset A of a topological group X is *Haar-null* if there exists a σ-additive Borel probability measure μ on X such that $\mu(A + x) = 0$ for all $x \in X$.

Theorem (Christensen, 1972)

A Borel subset A of a locally compact group X is Haar-null if and only if A has Haar measure zero.

Fact (folklore)

A Borel subset A of a Polish group X is not Haar-null if A is thick in the sense that for any compact set $K \subset X$ there exists $x \in X$ such that $K + x \subset A$.

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Theorem (Chistensen, 1972)

If a Borel subset A of a complete metric abelian group X is not Haar null, then $A - A$ is a neighborhood of zero in X.

This theorem shows that the difference part of Steinhaus-Weil Theorem generalizes to non-locally compact groups.
What about the sum part of the Steinhaus-Weil Theorem?

Example

The closed set $A := [0, +\infty)^\omega$ is not Haar null in the Polish group $X := \mathbb{R}^\omega$ and $A - A = X$. But $A + A = A$ is nowhere dense in X.

Yet, the set $A := [0, +\infty)^\omega$ is quite large in $\mathbb{R}^\omega$ – it contains shifts of all compact subsets of X, so A is thick.

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A subset A of a topological group X is called *locally thick* (or else *Haar-open*) if for any compact subset $K \subset X$ and point $p \in K$ there exists a neighborhood $O_p \subset K$ of x in K such that $O_p \subset A + x$ for some $x \in X$.

Proposition

A subset A of a Polish abelian group X is Haar-open if and only if for any compact set $K \subset X$ there exists $x \in X$ such that $K \cap (A + x)$ has non-empty interior in K.

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The set $A = [0, +\infty)^\omega$ is (locally) thick in the Polish group $\mathbb{R}^\omega$.

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Main Theorem

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Let $X := \prod_{n\in\omega} X_n$ be the countable product of Abelian locally compact topological groups. If Borel sets $A, B \subset X$ are not Haar-null in X, then the sum-set $A + B$ is Haar-open in X.

Property $(+)$

Definition

A topological group X is defined to have the property $(+)$ if for any Borel set A, B in X the sum-set $A + B$ is Haar-open in X if A, B are not Haar-null in X.

Main Theorem implies that the countable product of Abelian locally compact topological groups has the property $(+)$.

Problem

Which Abelian Polish groups have property $(+)$?

Conjecture (Bogachev)

A Fréchet (= locally convex complete metric vector) space X has property $(+)$ if and only if X is nuclear.

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T.Banakh, *Haar-open sets: a right way of generalizing the Steinhaus sum theorem to non-locally compact groups?*, preprint (https://arxiv.org/abs/1805.07515).

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